Lie algebras of smooth sections

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July 2007

Abstract

Lie algebras of smooth sections are Lie algebras obtained from bundles of Lie algebras, where the latter are vector bundles of which the fibers are Lie algebras. We also consider the C^k-sections for $k \in \mathbb{N}$. This paper, which is essentially my diploma thesis from May 2007 at Technische Universität Darmstadt, studies the derivations, the centroid and the isomorphisms of such Lie algebras and generalizes some facts from Lecomte's publications [Le79] and [Le80] to the case where the fiber is perfect or centerfree and it gives some more explicit proofs.

Contents

1 Introduction

1.1 Motivation and requirements

There are two main goals in the analysis of Lie algebras of smooth sections: On the one hand, the infinite-dimensional Lie algebras are not yet as well studied as the finite-dimensional ones and the methods used in finite-dimensional Lie theory are difficult to adapt to the infinite-dimensional case. I will discuss some properties of the Lie algebras of smooth sections in Chapter 3, especially their derivations, centroids and isomorphisms.

A solid knowledge of analysis, linear algebra, topology and some knowledge of the theory of manifolds, e.g. a priori to be acquired in [Ne05], is required. However, there is no knowledge of Lie algebra theory required, for the relevant parts will be explained.

1.2 Remarks concerning notation

I will follow some notational conventions which ought to be clarified.

- The formulae $A \subseteq B$ and $B \supseteq A$ will mean that each element of the set A is an element of the set B and equality is not excluded. In order to describe a subset relation excluding equality I will write $A \subsetneq B$ or $B \supsetneq A$.
- All vector spaces and (bi)linear maps are considered over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- The set $\{0, 1, 2, \ldots\}$ is denoted by $\mathbb{N}$ and the set $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is denoted by $\overline{\mathbb{N}}$.
- A C^k-map, $k \in \overline{\mathbb{N}}$, is a k-times continuously differentiable function. A function is also called "smooth" instead of C^∞.
- A manifold M will always be smooth, finite-dimensional over $\mathbb{R}$, hausdorff, paracompact and connected.
- Let $(e_1, \ldots, e_m)$ denote the canonical basis of $\mathbb{R}^m$. Then (U, ξ) being a chart of a manifold M with $\dim M = m$, $x \in U$ and $f : U \to E$ being a C^1-map between U and a finite-dimensional vector space E, we define the notation

$$\left(\partial_{\xi_i}\right)_x f := \partial_{\xi_i} f(x) := \left.\frac{d}{dt}\right|_{t=0} f\left(\xi^{-1}(\xi(x) + te_i)\right).$$

  As a basis of $T_x M$ we can take $\left(\left(\partial_{\xi_1}\right)_x, \ldots, \left(\partial_{\xi_m}\right)_x\right)$, where $\left(\partial_{\xi_i}\right)_x := T_{\xi(x)}\left(\xi^{-1}\right)(e_i)$. We write ∂_{ξ_i} for the map $U \to T_x M$, $x \mapsto \left(\partial_{\xi_i}\right)_x$. Any vector field $X \in \mathcal{V}(M)$ takes the local form $\sum_{i=1}^m X^i \partial_{\xi_i}$ for certain functions $X^1, \ldots X^m \in \mathrm{C}^\infty(U, \mathbb{R})$. For $f \in \mathrm{C}^\infty(M, \mathbb{R})$ we define $X.f \in \mathrm{C}^\infty(M, \mathbb{R})$ by $(X.f)(x) := (T_x f)(X_x)$ and we see that the two different definitions of $\left(\partial_{\xi_i}\right)_x$ are linked by the formula $\left(\partial_{\xi_i}\right)_x (f) = \left(\left(\partial_{\xi_i}\right).f\right)(x)$.

  If f is a C^k-function and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ a multi-index with $|\alpha| := \sum_{i=1}^m \alpha_i \leq k$, then we write:

$$\partial_\xi^\alpha f := \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_m}^{\alpha_m} f.$$

  For a multi-index $\alpha \in \mathbb{N}^m$ we also define $\alpha! := \prod_{i=1}^m \alpha_i$. Then, if $\xi(U)$ is convex, the Taylor Formula can be stated as follows for $x, y \in U$ and some $z \in [0,1] \cdot (x - y) + y \subseteq U$:

$$f(x+y) = \sum_{|\alpha|<k} \frac{\prod_{i=1}^m \xi_i(y)^{\alpha_i}}{\alpha!} \cdot \partial_\xi^\alpha f(x) + \sum_{|\alpha|=k} \frac{\prod_{i=1}^m \xi_i(y)^{\alpha_i}}{\alpha!} \cdot \partial_\xi^\alpha f(z).$$

  Furthermore, we define the notation $\gamma \leq \alpha$ for multi-indices $\alpha, \gamma \in \mathbb{N}^m$, which means $\gamma_i \leq \alpha_i$ for all $i \in \{1, \ldots, m\}$, and we define the multinomial coefficients for multi-indices $\gamma \leq \alpha \in \mathbb{N}^m$:

$$\binom{\alpha}{\gamma} := \frac{\alpha!}{\gamma!(\alpha - \gamma)!}.$$

  For multi-indices $\gamma \leq \alpha \in \mathbb{N}^m$ and any canonical basis vector e_i of $\mathbb{R}^m$, the multinomial coefficients satisfy the formula

$$\binom{\alpha + e_i}{\gamma + e_i} = \binom{\alpha}{\gamma} + \binom{\alpha}{\gamma + e_i}.$$

- A Lie group is a manifold equipped with a smooth group multiplication whose inversion is smooth.
- The neutral element of a group is denoted by 1 and if g is an element of the group, then λ_g is the multiplication by g from the left and ρ_g is the multiplication by g from the right.

- Depending on the context, the symbol $\mathbf{1}$ denotes the identity map of a vector space, the endomorphism of a vector bundle where each $\mathbf{1}_x$ is the identity of the x-fiber, or a constant map with value 1 on a manifold.
- For numbers $m, n \in \mathbb{N} \setminus \{0\}$, the space of all $m \times n$-matrices with entries in $\mathbb{K}$ is denoted by $M_{m,n}(\mathbb{K})$ and is canonically identified with the space of linear functions $\mathbb{K}^n \to \mathbb{K}^m$. Note that, if we have $m = 0$ or $n = 0$, then $M_{m,n}(\mathbb{K})$ is the vector space containing the unique linear map $\mathbb{K}^n \to 0$ or $0 \to \mathbb{K}^n$, respectively.
- If $f : M \to E$ is a C^k-map between a manifold and a finite-dimensional vector space, $n \in \mathbb{N}$ with $n \leq k$ and $x \in M$, then we define the notation
$$j_x^n(f) = 0 \quad :\Longleftrightarrow \quad \begin{cases} f(x) = 0 & \text{if } n = 0 \\ j_x^{n-1}(f) = 0 \text{ and } T_x^n f = 0 & \text{if } n > 0. \end{cases}$$
- If, for each point $x \in M$ of a set, $T_x = T(x) : N \to P$ is a map and $A_x = A(x) \in N$ is a point, then we denote the map $M \to P$, $x \mapsto T_x(A_x)$ by $T \cdot A$. If, for each point $x \in M$ of a set, $T_x = T(x) : N \to P$ is a map and $S_x = S(x) : L \to N$ is a map, then we denote the map $M \to P^L$, $x \mapsto T_x \circ S_x$ by $T \circ S$.

2 Definitions, notions, former results

2.1 Associative algebras and Lie algebras

Definition 2.1. Let A be a vector space.

1. A is called an *algebra* if equipped with a bilinear map $A \times A \to A, (x, y) \mapsto x \cdot y$, which we call the *multiplication* of the algebra A.
2. An algebra A is called *associative*, if $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ is satisfied for all $x, y, z \in A$. In this case the expression $x \cdot y \cdot z$ is clear without ambiguity. An associative algebra A may possess an element $1 \in A$ with $1 \cdot x = x \cdot 1 = x$ for all $x \in A$, called *unity element*.
3. An algebra $\mathfrak{g}$ is called a *Lie algebra* and its multiplication map is also written $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$, if the following two conditions are satisfied:
   (a) $[\cdot, \cdot]$ is alternating, i.e. $[x, x] = 0$ for all $x \in \mathfrak{g}$.
   (b) the *Jacobi identity*, i.e. $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$ for all $x, y, z \in \mathfrak{g}$.
4. If S, T are subsets of an algebra A, we define $S \cdot T$ to be the vector space generated by the set $\{s \cdot t : s \in S, t \in T\}$. In the Lie algebra case we also write $[S, T]$ instead of $S \cdot T$.
5. Let B be a vector subspace of an algebra A. It is called *left ideal* of A, if $A \cdot B \subseteq B$, i.e. $x \cdot y \in B$ for all $x \in A$, $y \in B$. It is called *right ideal* of A, if $B \cdot A \subseteq B$, i.e. $y \cdot x \in B$ for all $x \in A$, $y \in B$. It is called *(two-sided) ideal* of A, symbolically $B \trianglelefteq A$, if it is a left ideal and a right ideal of A. In the Lie algebra case, the conditions $A \cdot B \subseteq B$ and $B \cdot A \subseteq B$ are equivalent.
6. A vector subspace B of an algebra A is called a *subalgebra* of A, symbolically $B \leq A$, if $x \cdot y \in B$ for all $x, y \in B$, i.e. the multiplication of A induces a multiplication of B. Of course a subalgebra of A is an algebra, too.
7. Let $\mathfrak{g}$ be a Lie algebra. The vector subspace $[\mathfrak{g}, \mathfrak{g}]$ is called the *commutator* of $\mathfrak{g}$ and $\mathfrak{g}$ is *perfect* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. The vector subspace $\{ x \in \mathfrak{g} | [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$ is called the *center* of $\mathfrak{g}$, denoted by $\mathfrak{z}(\mathfrak{g})$, and $\mathfrak{g}$ is *abelian* if $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}$ or, equivalently, $[\mathfrak{g}, \mathfrak{g}] = 0$. The Lie algebra $\mathfrak{g}$ is called *centerfree*, if $\mathfrak{z}(\mathfrak{g}) = 0$.

8. A Lie algebra $\mathfrak{g}$ is *nilpotent*, if its *lower central series* $\mathfrak{g}^0, \mathfrak{g}^1, \mathfrak{g}^2, \ldots$, which is defined by $\mathfrak{g}^0 := \mathfrak{g}$ and $\mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$ for $n \in \mathbb{N}$, becomes zero eventually.

9. A Lie algebra $\mathfrak{g}$ is *solvable*, if its *derived series* $\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \ldots$, which is defined by $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(n+1)} := \left[\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}\right]$ for $n \in \mathbb{N}$, becomes zero eventually.

Remark 2.2. In our case of $\mathbb{K}$ being a field of characteristic 0, an algebra multiplication is alternating if and only if it is skew-symmetric: On the one hand we have, for all $x, y \in \mathfrak{g}$:

$$[x,y] = -[y,x] \implies [x,x] = -[x,x] \implies 2[x,x] = 0 \implies [x,x] = 0.$$

On the other hand, for all $x, y, z \in \mathfrak{g}$:

$$\begin{aligned} [z,z] = 0 &\implies 0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x] \\ &\implies [x,y] = -[y,x]. \end{aligned}$$

The next lemma (of which I leave out the elementary proof) shows four “canonical” ways to construct algebras.

Lemma 2.3.

1. *Let V be an arbitrary vector space. Then its vector space endomorphisms form an associative algebra where the multiplication of the algebra is $\circ$, the composition of functions. This associative algebra is called the* endomorphism algebra *of V, denoted by* $\mathrm{End}(V)$[1].

2. *Let A be an associative algebra. A equipped with the map $[\cdot,\cdot] : A \times A \to A$, $(a,b) \mapsto a \cdot b - b \cdot a$, called the* commutator bracket *of A, is a Lie algebra, denoted by A_L.*

3. *A being an algebra, the vector space $\{ f \in \mathrm{End}(A) |\, f(a \cdot b) = f(a) \cdot b + a \cdot f(b)$ for all $a, b \in A\}$ is a Lie subalgebra of $\mathrm{End}(A)_L$, called the* Lie algebra of derivations *of A, denoted by* $\mathrm{Der}(A)$.

4. *A being an algebra and $I \trianglelefteq A$ an ideal, the quotient vector space A/I equipped with the well-defined induced multiplication*

$$\begin{aligned} A/I \times A/I &\longrightarrow A/I \\ (a+I, b+I) &\longmapsto (a \cdot b) + I \end{aligned}$$

is an algebra, called quotient algebra *of A modulo I, denoted by A/I.*

Definition 2.4. The set of *Lie algebra morphisms* of $\mathfrak{g}$ and $\mathfrak{h}$ is

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{h}) := \{ f : \mathfrak{g} \to \mathfrak{h} \text{ linear map} \mid f([x,y]) = [f(x), f(y)] \text{ for all } x, y \in \mathfrak{g} \}.$$

The set of *Lie algebra isomorphimsm* of $\mathfrak{g}$ and $\mathfrak{h}$ is

$$\mathrm{Iso}(\mathfrak{g}, \mathfrak{h}) := \{ f : \mathfrak{g} \to \mathfrak{h} \text{ linear isomorphism} \mid f([x,y]) = [f(x), f(y)] \text{ for all } x, y \in \mathfrak{g} \}.$$

In the special case of $\mathfrak{g} = \mathfrak{h}$ we even have a group $\mathrm{Aut}(\mathfrak{g}) := \mathrm{Iso}(\mathfrak{g}, \mathfrak{g})$ with neutral element $\mathbf{1} = \mathrm{id}_{\mathfrak{g}}$, which is called *group of Lie algebra automorphisms* of $\mathfrak{g}$.

Now it is time to give some important examples of algebras.

Example 2.5.

1. Every vector space V can be equipped with the *trivial Lie bracket*: $[x,y] := 0$ for all $x, y \in V$.

[1]The symbol $\mathrm{End}(V)$ will also be used to describe only the *set* of vector space endomorphisms of V.

2. S being a set and A being an algebra, the set A^S of all mappings $S \to A$ can be pointwisely equipped with an algebra structure. An important subalgebra of such an algebra is $\mathrm{C}^k(M, \mathbb{K}) \leq \mathbb{K}^M$, where M is a smooth manifold, $\mathbb{K}$ is equipped with the field multiplication as algebra multiplication and $k \in \overline{\mathbb{N}}$.

3. If $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra and $\varphi : V \to \mathfrak{g}$ is an isomorphism of vector spaces, then we can define a Lie bracket on V as follows:

$$\begin{aligned} [\cdot,\cdot]_\varphi : V \times V &\longrightarrow V \\ (v, w) &\longmapsto \varphi^{-1}\left[\varphi(v), \varphi(w)\right]. \end{aligned}$$

   Then φ becomes an isomorphism of the Lie algebras $\left(V, [\cdot,\cdot]_\varphi\right)$ and $(\mathfrak{g}, [\cdot,\cdot])$.

4. If V is a vector space, we call $\mathfrak{gl}(V) := \mathrm{End}(V)_L$ the *general Lie algebra of* V.

5. If V is a vector space and $\dim(V) < \infty$, we call $\mathfrak{sl}(V) := \{\, f \in \mathrm{End}(V) |\, \mathrm{tr}(f) = 0\}$ the *special Lie algebra of* V. In fact, $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$:

$$\mathrm{tr}\left([f, g]\right) = \mathrm{tr}(f \circ g - g \circ f) = \mathrm{tr}(f \circ g) - \mathrm{tr}(g \circ f) = \mathrm{tr}(f \circ g) - \mathrm{tr}(f \circ g) = 0$$

   This calculation even shows that $[\mathfrak{gl}(V), \mathfrak{gl}(V)] \subseteq \mathfrak{sl}(V)$.

6. The squared matrices over $\mathbb{K}$ with n rows form the associative algebra $M_{n,n}(\mathbb{K})$ with respect to the multiplication of matrices. It is isomorphic to $\mathrm{End}\left(\mathbb{K}^n\right)$. The corresponding Lie algebra $\left(M_{n,n}(\mathbb{K})\right)_L$ isomorphic to $\mathfrak{gl}\left(\mathbb{K}^n\right)$ is denoted by $\mathfrak{gl}_n(\mathbb{K})$ and called *general linear Lie algebra of order* n.

   Analogously, $\mathfrak{sl}_n(\mathbb{K}) := \{\, A \in M_n(\mathbb{K}) |\, \mathrm{tr}(A) = 0\}$ is called *special linear Lie algebra of order* n and is isomorphic to $\mathfrak{sl}(\mathbb{K}^n)$. We identify $\mathrm{End}\left(\mathbb{K}^n\right), \mathfrak{gl}\left(\mathbb{K}^n\right), \mathfrak{sl}\left(\mathbb{K}^n\right)$ with $M_n(\mathbb{K})$, $\mathfrak{gl}_n(\mathbb{K})$, $\mathfrak{sl}_n(\mathbb{K})$, respectively.

7. Let $\beta : V \times V \to \mathbb{K}$ be a bilinear map. The set

$$\mathfrak{o}(V, \beta) := \{\, f \in \mathrm{End}(V) |\, \beta(f(x), y) + \beta(x, f(y)) = 0\}$$

   is a $\mathbb{K}$-Lie subalgebra of $\mathfrak{gl}(V)$, called the *Lie algebra of β-skew-symmetric endomorphisms* of V. There are some important examples of this type of Lie algebra:

   (a) For $V = \mathbb{K}^n$ and $\beta(x, y) = \sum_{i=1}^n x_i y_i$ we write $\mathfrak{o}_n(\mathbb{K}) := \mathfrak{o}(V, \beta)$. This is the *orthogonal Lie algebra of order* n, which can be identified with

$$\left\{ A \in M_n(\mathbb{K}) |\, A + A^T = 0 \right\},$$

   the set of skew-symmetric $n \times n$-matrices. The space $\mathfrak{so}_n(\mathbb{K}) := \mathfrak{o}_n(\mathbb{K}) \cap \mathfrak{sl}_n(\mathbb{K})$ is a Lie subalgebra of $\mathfrak{o}_n(\mathbb{K})$, called *special orthogonal Lie algebra of order* n.[2]

   (b) For $V = \mathbb{C}^n$ and $\beta(x, y) = \sum_{i=1}^n x_i \overline{y_i}$ we write $\mathfrak{u}_n(\mathbb{C}) := \mathfrak{o}(V, \beta)$. This is the *unitary Lie algebra of order* n, which can be identified with

$$\left\{ A \in M_n(\mathbb{C}) |\, A + \overline{A^T} = 0 \right\},$$

   the set of complex skew-hermitian $n \times n$-matrices. Note that this is not a complex, but a real Lie algebra, since it is not a $\mathbb{C}$-vector space. The space $\mathfrak{su}_n(\mathbb{C}) := \mathfrak{u}_n(\mathbb{C}) \cap \mathfrak{sl}_n(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{u}_n(\mathbb{C})$, called *special unitary Lie algebra of order* n.

[2] In our considered case of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, there is no difference between $\mathfrak{so}_n(\mathbb{K})$ and $\mathfrak{o}_n(\mathbb{K})$. The case $\mathfrak{so}_n(\mathbb{K}) \subsetneq \mathfrak{o}_n(\mathbb{K})$ is only possible if $\mathrm{char}(\mathbb{K}) = 2$.

(c) For $V = \mathbb{K}^{2n}$ and $\beta(x,y) = \sum_{i=1}^{n} x_i y_{n+i} - x_{n+i} y_i$ we write $\mathfrak{sp}_{2n}(\mathbb{K}) := \mathfrak{o}(V,\beta)$. This is the *symplectic Lie algebra of order* n, which can be identified with

$$\left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in M_2\left(M_n(\mathbb{K})\right) \cong M_{2n}(\mathbb{K}) \middle| B = B^T, C = \mathrm{C}^T \right\}.$$

Remark 2.6.

1. One can show that for any Lie algebra $\mathfrak{g}$ there is an associative algebra with unity $\mathcal{U}(\mathfrak{g})$, such that $\mathfrak{g}$ can be embedded into $\left(\mathcal{U}(\mathfrak{g})\right)_L$ via an injective morphism of Lie algebras $\eta_{\mathfrak{g}} : \mathfrak{g} \to \left(\mathcal{U}(\mathfrak{g})\right)_L$ (cf. Part III of [Ne94]). If one demands a universal property[3] of $\mathcal{U}(\mathfrak{g})$ and $\eta_{\mathfrak{g}}$, then $(\mathcal{U}(\mathfrak{g}), \eta_{\mathfrak{g}})$ is unique up to algebra isomorphism, i.e. an isomorphism of vector spaces preserving the multiplications, and $(\mathcal{U}(\mathfrak{g}), \eta_{\mathfrak{g}})$ is called *universal enveloping algebra* of $\mathfrak{g}$.

2. The *Ado Theorem* states that each finite-dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$ for certain $n \in \mathbb{N}$, i.e. each finite-dimensional Lie algebra is, up to isomorphism, a Lie algebra of squared matrices.

Definition 2.7. Let $\mathfrak{g}$ be a Lie algebra.

1. If V is a vector space, then a *representation of* $\mathfrak{g}$ *on* V is a morphism of Lie algebras

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

2. A $\mathfrak{g}$-*module* is a vector space V with a $\mathbb{K}$-bilinear map

$$\begin{aligned} \mu : \mathfrak{g} \times V &\longrightarrow V \\ (x,v) &\longmapsto \mu(x,v) =: x.v \end{aligned}$$

satisfying the equation

$$[x,y].v = x.(y.(v)) - y.(x.(v))$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

It is not difficult to obtain a bijective correspondence between the concepts of $\mathfrak{g}$-module and representation of $\mathfrak{g}$: Given a module with multiplication $\mu : \mathfrak{g} \times V \to V$ one can define a representation by

$$\begin{aligned} r(\mu) : \mathfrak{g} &\longrightarrow \mathfrak{gl}(V) \\ x &\longmapsto (v \mapsto \mu(x,v)). \end{aligned}$$

Given a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ one can define a module structure on V by

$$\begin{aligned} m(\rho) : \mathfrak{g} \times V &\longrightarrow V \\ (x,v) &\longmapsto \rho(x)(v). \end{aligned}$$

It is easy to verify that $r(m(\rho)) = \rho$ and $m(r(\mu)) = \mu$ for every representation ρ and every module multiplication μ.

A linear map $f : V \to W$ between two modules is called *module morphism*, if $f(x.v) = x.f(v)$ for all $x \in \mathfrak{g}, v \in V$. The vector space of module morphisms $V \to W$ is denoted by $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ and we write $\mathrm{End}_{\mathfrak{g}}(V) := \mathrm{Hom}_{\mathfrak{g}}(V,V)$. A vector subspace $W \subseteq V$ of a module which is invariant under $\mathfrak{g}$ is itself a module, called *submodule*. Images and kernels of module morphisms, quotients of modules modulo submodules and intersections of submodules are again modules. Note that the direct sum (in the sense of vector subspaces) of two submodules is also a submodule.

[3] If A is an associative algebra with unity and $\alpha : \mathfrak{g} \to A_L$ is a morphism of Lie algebras, then there is a unique morphism of associative algebras $\alpha' : \mathcal{U}(\mathfrak{g}) \to A$ such that $\alpha'(1) = 1$ and $\alpha' \circ \eta_{\mathfrak{g}} = \alpha$.

3. A module $V \neq 0$ is called
   (a) *simple*, if 0 is its only proper submodule.
   (b) *semisimple*, if for each of its submodules W there is a complementary submodule $W' \subseteq V$ such that $W \oplus W' = V$.
   (c) *trivial*, if $\mathfrak{g}.V = 0$.

Some useful facts about simple and semisimple modules are presented in the following two lemmas, shown e.g. in [Ne02]:

Lemma 2.8.

1. *Submodules and quotient modules of semisimple modules are semisimple, too.*
2. *The following statements are equivalent for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V:*
   (a) V is semisimple.
   (b) V is a sum of simple modules.
   (c) V is a direct sum of simple modules.

Lemma 2.9. (Schur Lemma) *Let V, W be simple $\mathfrak{g}$-modules of a Lie algebra $\mathfrak{g}$.*

1. $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$, *if V is not isomorphic to W.*
2. *Each non-zero element of* $\mathrm{End}_{\mathfrak{g}}(V)$ *is invertible.*
3. *If* $\dim(V) < \infty$ *and* $\mathbb{K} = \mathbb{C}$, *then* $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C} \cdot \mathbf{1}$.

Definition 2.10. Let $\mathfrak{g}$ be a Lie algebra. We define the *adjoint representation*[4] of $\mathfrak{g}$ to be the map

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathrm{Der}(\mathfrak{g}) \\ x &\longmapsto \mathrm{ad}_x := (y \mapsto [x, y]) . \end{aligned}$$

The Jacobi identity for $\mathfrak{g}$ ensures that the range is properly chosen. Note that $\mathfrak{z}(\mathfrak{g}) = \ker(\mathrm{ad})$.

The adjoint representation of $\mathfrak{g}$ corresponds to the $\mathfrak{g}$-module structure on $\mathfrak{g}$, where the multiplication is the Lie bracket of $\mathfrak{g}$. In this sense the ideals of $\mathfrak{g}$ are the submodules of $\mathfrak{g}$. We use this change of perspective in order to define the notions “simple” and “semisimple” for Lie algebras.

Definition 2.11. Let $\mathfrak{g}$ be a Lie algebra and consider it as a $\mathfrak{g}$-module with respect to the adjoint representation.

1. $\mathfrak{g}$ is a *simple* Lie algebra, if it is a *non-trivial* simple $\mathfrak{g}$-module.
2. $\mathfrak{g}$ is a *semisimple* Lie algebra, if it is the direct sum of *non-trivial* simple $\mathfrak{g}$-modules, i.e. of simple Lie algebras.
3. $\mathfrak{g}$ is a *reductive* Lie algebra, if it is the direct sum of simple $\mathfrak{g}$-modules., i.e. ideals of $\mathfrak{g}$.

Example 2.12. The Lie algebras $\mathfrak{sl}_n(\mathbb{K})$ for $n \geq 2$ and $\mathfrak{so}_n(\mathbb{K}) = \mathfrak{o}_n(\mathbb{K})$ for $n \geq 5$ and $\mathfrak{sp}_{2n}(\mathbb{K})$ for $n \geq 1$ are simple. The Lie algebras $\mathfrak{gl}_n(\mathbb{K})$ for $n \geq 1$ are reductive and $\dim \mathfrak{z}\left(\mathfrak{gl}_n(\mathbb{K})\right) = 1$. For proofs, cf. e.g. [Ne02].

Remark 2.13. If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a decomposition of a Lie algebra into a direct sum of ideals, then $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{a} \cap \mathfrak{b} = 0$.

[4] ad represents $\mathfrak{g}$ on $\mathfrak{g}$: $\mathrm{ad}_{[x,y]}(z) = [[x, y], z] = -[y, [x, z]] + [x, [y, z]] = [\mathrm{ad}_x, \mathrm{ad}_y](z)$
$\Longrightarrow$ $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$.

The relation between reductive and semisimple Lie algebras is described in the following lemma. For a proof, cf. e.g. [Ne02].

Lemma 2.14. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If $\mathfrak{g}$ is semisimple, then it is reductive, perfect, centerfree and* $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$. *So* $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ *is an isomorphism of Lie algebras.*

2. *If $\mathfrak{g}$ is reductive, then $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$ and $[\mathfrak{g},\mathfrak{g}]$ is semisimple. In particular, a reductive Lie algebra is semisimple if and only if it is centerfree if and only if it is perfect.*

Example 2.15. Let $\mathfrak{g}$ be a two-dimensional $\mathbb{K}$-vector space with basis (x_1, x_2). The unique skew-symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ determined by the relation $[x_1, x_2] := x_1$ is a Lie bracket. The Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is centerfree, but not perfect, thus not reductive. Let $\mathfrak{h}$ be a five-dimensional $\mathbb{K}$-vector space with basis $(y_1, \ldots, y_5)$. The unique skew-symmetric bilinear map $\{\cdot,\cdot\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ determined by the relations

$$\{y_1, y_2\} := y_1, \quad \{y_1, y_3\} := y_2, \quad \{y_1, y_4\} := y_3, \quad \{y_2, y_3\} := y_4, \quad \{y_2, y_4\} := y_5,$$
$$\{y_1, y_5\} := \{y_2, y_5\} := \{y_3, y_4\} := \{y_3, y_5\} := \{y_4, y_5\} := 0.$$

is a Lie bracket. The Lie algebra $(\mathfrak{h}, \{\cdot,\cdot\})$ possesses the non-zero center $\mathbb{K} \cdot y_5$, but it is perfect, thus not reductive.

An interesting vector subspace of $\mathrm{End}(\mathfrak{g})$ is the centroid of $\mathfrak{g}$.

Definition 2.16. The *commutant of the adjoint representation* or *centroid* of a Lie algebra $\mathfrak{g}$ is defined as follows:

$$\begin{aligned}\mathrm{Cent}(\mathfrak{g}) : &= \{ f \in \mathrm{End}(\mathfrak{g}) |\, [f, \mathrm{ad}_x] = 0 \text{ for all } x \in \mathfrak{g}\} = \{ f \in \mathrm{End}(\mathfrak{g}) |\, f \circ \mathrm{ad}_x = \mathrm{ad}_x \circ f \text{ for all } x \in \mathfrak{g}\} \\ &= \{ f \in \mathrm{End}(\mathfrak{g}) |\, f\,[x,y] = [x, f(y)] \text{ for all } x, y \in \mathfrak{g}\}.\end{aligned}$$

Note that for $f \in \mathrm{Cent}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$ we have:

$$[f(x), y] = -\,[y, f(x)] = -f\,[y,x] = f\,[x,y] = [x, f(y)]\,.$$

For a finer analysis of the structure of $\mathrm{Cent}(\mathfrak{g})$, we introduce special elements in an endomorphism algebra.

Definition 2.17. Let $\mathrm{End}(V)$ be the endomorphism algebra of a vector space V and $\mathfrak{g}$ a Lie algebra.

1. A map $f \in \mathrm{End}(V)$ is *nilpotent*, if there exists $n \in \mathbb{N}$ such that $f^n = 0$.[5] A map $f \in \mathrm{End}(V)$ is *semisimple*, if for each f-invariant subspace $W \subseteq V$ there exists a complementary f-invariant subspace $W' \subseteq V$ such that $W \oplus W' = V$.

2. We define the subsets $\mathrm{N}(\mathfrak{g}) := \{ f \in \mathrm{Cent}(\mathfrak{g}) |\, f \text{ nilpotent}\}$, $\mathrm{S}(\mathfrak{g}) := \{ f \in \mathrm{Cent}(\mathfrak{g}) |\, f \text{ semisimple}\}$ and the vector subspace $\mathrm{J}(\mathfrak{g}) := \{ \varphi \in \mathrm{Cent}(\mathfrak{g}) |\, \mathrm{ad}_x \circ \varphi = 0 = \varphi \circ \mathrm{ad}_x \text{ for all } x \in \mathfrak{g}\}$.

Lemma 2.18. *Let $\mathfrak{g}$ be a Lie algebra.*

1. $\mathrm{Cent}(\mathfrak{g})$ *is an associative subalgebra of* $\mathrm{End}(\mathfrak{g})$.

2. *We have the inclusions* $\mathrm{Cent}(\mathfrak{g}) \circ \mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$ *and* $[\mathrm{Cent}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g})] \subseteq \mathrm{Cent}(\mathfrak{g})$ *or, to say it in other words,* $\mathrm{Der}(\mathfrak{g})$ *is a* $\mathrm{Cent}(\mathfrak{g})$*-module and* $\mathrm{Der}(\mathfrak{g})$ *acts by module derivations on* $\mathrm{Cent}(\mathfrak{g})$.

3. *We have the inclusion* $[\mathrm{Cent}(\mathfrak{g}), \mathrm{Cent}(\mathfrak{g})] \subseteq \mathrm{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}], \mathfrak{z}(\mathfrak{g}))$. *In particular, if $\mathfrak{g}$ is perfect or centerfree, then* $\mathrm{Cent}(\mathfrak{g})$ *is abelian. In the latter case, if $\mathfrak{g}$ is of finite dimension, then* $\mathrm{N}(\mathfrak{g}), \mathrm{S}(\mathfrak{g})$ *are associative subalgebras of* $\mathrm{Cent}(\mathfrak{g})$ *with* $\mathrm{Cent}(\mathfrak{g}) = \mathrm{N}(\mathfrak{g}) \oplus \mathrm{S}(\mathfrak{g})$.

[5] Note that in the case of $\dim V = d < \infty$ this is equivalent to $f^d = 0$.

4. *If $\mathfrak{g}$ is of dimension $d \in \mathbb{N}$ and semisimple, then* $\mathrm{N}(\mathfrak{g}) = 0$, *thus* $\mathrm{Cent}(\mathfrak{g}) = \mathrm{S}(\mathfrak{g})$.

Proof.

1. Clearly, $\mathrm{Cent}(\mathfrak{g})$ is a vector subspace of $\mathrm{End}(\mathfrak{g})$. If $f, g \in \mathrm{Cent}(\mathfrak{g})$, then for all $x, y \in \mathfrak{g}$ we have:

$$(fg)[x,y] = f(g[x,y]) = f([x,g(y)]) = [x, f(g(y))] = [x, (fg)(y)].$$

So, fg is also in $\mathrm{Cent}(\mathfrak{g})$. Therefore $\mathrm{Cent}(\mathfrak{g})$ is an associative subalgebra of $\mathrm{End}(\mathfrak{g})$.

2. Let $f \in \mathrm{Cent}(\mathfrak{g})$, $g \in \mathrm{Der}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$. Then:

$$\begin{aligned}[f,g][x,y] &= f(g[x,y]) - g(f[x,y]) = f([gx,y] + [x,gy]) - g([x,fy]) \\ &= f([gx,y]) + f([x,gy]) - [gx,fy] - [x,gfy] = [gx,fy] + [x,fgy] - [gx,fy] - [x,gfy] \\ &= [x,fgy] - [x,gfy] = [x, fgy - gfy] = [x,[f,g](y)].\end{aligned}$$

This implies $[\mathrm{Cent}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g})] \subseteq \mathrm{Cent}(\mathfrak{g})$. We may also calculate:

$$(fg)[x,y] = f([gx,y] + [x,gy]) = [(fg)x, y] + [x, (fg)y].$$

We thus get $\mathrm{Cent}(\mathfrak{g}) \circ \mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$.

3. Let $f, g \in \mathrm{Cent}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$. We calculate:

$$\begin{aligned}[[f,g](x), y] &= [f(gx) - g(fx), y] = f[gx,y] - g[fx,y] = fg[x,y] - gf[x,y] \\ &= [fx, gy] - [fx, gy] = 0.\end{aligned}$$

Therefore $[f,g](\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$. If $z \in [\mathfrak{g}, \mathfrak{g}]$ is arbitrarily chosen, then there exist finitely many elements $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathfrak{g}$ such that $z = \sum_{i=1}^{n} [x_i, y_i]$ and thus we obtain

$$[f,g](z) = \sum_{i=1}^{n} fg[x_i, y_i] - gf[x_i, y_i] = 0$$

implying $[f,g]([\mathfrak{g},\mathfrak{g}]) = 0$ and so $[f,g]$ identifies with a vector space morphism $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathfrak{z}(\mathfrak{g})$.

Now let $\mathrm{Cent}(\mathfrak{g})$ be abelian and $\mathfrak{g}$ of finite dimension. Obviously, the subsets $\mathrm{N}(\mathfrak{g}), \mathrm{S}(\mathfrak{g})$ are closed under the multiplication by scalars in $\mathbb{K}$.

Let $f, g \in \mathrm{N}(\mathfrak{g})$ and $n, m \in \mathbb{N}$ such that $f^n = g^m = 0$. Obviously, $f \circ g$ is also nilpotent. Since $\mathrm{Cent}(\mathfrak{g})$ is abelian, we have $[f,g] = 0$ and so we can apply the Binomial Theorem, yielding

$$\begin{aligned}(f+g)^{n+m+1} &= \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} f^k g^{n+m-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n+m-1}{n-1-k} f^{n-1-k} g^{m+k} + \sum_{k=0}^{m-1} \binom{n+m-1}{n+k} f^{n+k} g^{m-k-1} = 0 + 0 = 0.\end{aligned}$$

Thus $f+g$ is also nilpotent. So $\mathrm{N}(\mathfrak{g})$ is an associative subalgebra of $\mathrm{Cent}(\mathfrak{g})$.

Let $f, g \in \mathrm{S}(\mathfrak{g})$. We want to show that $f+g$ and $f \circ g$ are also semisimple. If $\mathbb{K} = \mathbb{R}$, then we use Corollary V.2.8 of [Ne94] to say that $h \in \mathrm{End}(\mathfrak{g})$ would be semisimple if and only if its complexification $h_{\mathbb{C}} := \mathrm{id}_{\mathbb{C} \otimes \mathfrak{g}} \otimes h$ was semisimple. So we only consider the case $\mathbb{K} = \mathbb{C}$. By Lemma V.3.3 of [Ne94], "semisimple" means the same as "diagonalizable" for $\mathbb{K} = \mathbb{C}$. Since $\mathrm{Cent}(\mathfrak{g})$ is abelian, we have $[f,g] = 0$ and so we can apply a theorem about simultaneous diagonalzation (cf. e.g. Theorem 11.B.15 of [StWi90]): Diagonalizable endomorphisms of a vector space of finite dimension are simultaneously diagonalizable if and only if they commute. So f and g are diagonal

with respect to a fixed basis, thus also $f + g$ and $f \circ g$, so they are semisimple, too. This shows that $\mathrm{S}(\mathfrak{g})$ is an associative subspace of $\mathrm{Cent}(\mathfrak{g})$.

The Jordan Theorem V.3.5 of [Ne94] yields that each element $f \in \mathrm{Cent}(\mathfrak{g})$ decomposes into the sum of a nilpotent vector space endomorphism f_n and a semisimple one f_s and because of the fact that f commutes with any ad_x for $x \in \mathfrak{g}$ this is also satisfied for f_n and f_s. Since $\mathrm{Cent}(\mathfrak{g})$ is abelian, any two endomorphisms n, s with $f = n + s$, where $n \in \mathrm{N}(\mathfrak{g})$ and $s \in \mathrm{S}(\mathfrak{g})$, commute with f and so the theorem also implies that $n = f_n$ and $s = f_s$. This shows $\mathrm{Cent}(\mathfrak{g}) = \mathrm{N}(\mathfrak{g}) \oplus \mathrm{S}(\mathfrak{g})$.

4. If $f \in \mathrm{N}(\mathfrak{g})$, then $f^d(\mathfrak{g}) = 0$. On the other hand, $f(\mathfrak{g})$ is an ideal of $\mathfrak{g}$ because for $x, y \in \mathfrak{g}$ we have $[y, f(x)] = f\,[x, y] \in f(\mathfrak{g})$. Since $f(\mathfrak{g})^n \subseteq f^n(\mathfrak{g})$ for all $n \in \mathbb{N}$, this implies $f(\mathfrak{g})^d = 0$, so $f(\mathfrak{g})$ is a nilpotent ideal of $\mathfrak{g}$. But $\mathfrak{g}$ is semisimple, so this ideal is trivial and $f = 0$.

□

Remark 2.19.

1. Note that any finite-dimensional associative commutative $\mathbb{C}$-algebra A with unity element 1 can be realized as the centroid of a perfect and centerfree $\mathbb{C}$-Lie algebra $\mathfrak{g}$. We show this in several steps.

   (a) Let $\mathfrak{k}$ be an finite-dimensional simple $\mathbb{C}$-Lie algebra, e.g. $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{C})$. Then it is central, i.e.

   $$\mathrm{Cent}(\mathfrak{k}) = \mathrm{End}_{\mathfrak{k}}(\mathfrak{k}) = \mathbb{K} \cdot \mathbf{1},$$

   by the Schur Lemma. We define $\mathfrak{g} := \mathfrak{k} \otimes A$, understood as the tensor product of $\mathbb{K}$-vector spaces. By setting $[(x \otimes a), (y \otimes b)] := [x, y] \otimes ab$ for all generating elements $x, y \in \mathfrak{k}$ and $a, b \in A$ and extending $[\cdot, \cdot]$ to a $\mathbb{C}$-bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we obtain a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$.

   (b) Since $\mathfrak{k}$ is simple, it is perfect. This implies that $\mathfrak{g}$ is also perfect: An element $v \in \mathfrak{g}$ takes the form $v = \sum_{i=1}^n x_i \otimes a_i$ for certain $x_1, \ldots, x_n \in \mathfrak{k}$ and $a_1, \ldots, a_n \in A$ and each x_i can be written as $x_i = \sum_{j=1}^{r_i} [y_{ij}, z_{ij}]$ for some $y_{i1}, z_{i1}, \ldots, y_{ir_i}, z_{ir_i} \in \mathfrak{k}$. So we have

   $$v = \sum_{i=1}^{n} \left(\sum_{j=1}^{r_i} [y_{ij}, z_{ij}] \right) \otimes a_i = \sum_{i=1}^{n} \sum_{j=1}^{r_i} [y_{ij} \otimes a_i, z_{ij} \otimes 1]\,.$$

   (c) Furthermore, $\mathfrak{g}$ is centerfree because $\mathfrak{k}$ is so: If $v = \sum_{j=1}^m x_j \otimes a_j \in \mathfrak{z}(\mathfrak{g})$ for some $x_1, \ldots, x_m \in \mathfrak{k}$ and $a_1, \ldots, a_m \in A$, then, for all $x \in \mathfrak{k}$, we have:

   $$0 = [v, x \otimes 1] = \sum_{j=1}^{m} [x_j \otimes a_j, x \otimes 1] = \sum_{j=1}^{m} [x_j, x] \otimes a_j.$$

   But this is only possible, if, for all $j \in \{1, \ldots, m\}$, we have $a_j = 0$ or $x_j \in \mathfrak{z}(\mathfrak{k}) = 0$, thus $v = 0$.

   (d) We may consider $\mathfrak{k} \subseteq \mathfrak{g}$ by the embedding $\mathfrak{k} \hookrightarrow \mathfrak{g}$, $x \mapsto x \otimes 1$ and $\mathfrak{g}$ becomes a $\mathfrak{k}$-module by restriction of the adjoint representation. By [Ne02], there are $x_1, \ldots, x_n, x^1, \ldots, x^n \in \mathfrak{k}$ such that $\sum_{i=1}^n \mathrm{ad}(x_i) \circ \mathrm{ad}(x^i)$ is an element in $\mathrm{Cent}(\mathfrak{k})$ and, by [Hu72] on page 122, it is equal to $\lambda \cdot \mathbf{1}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and so we may assume, after normalizing:

   $$\sum_{i=1}^{n} \mathrm{ad}(x_i) \circ \mathrm{ad}(x^i) = \mathbf{1}.$$

   If $a \in A$ is an arbitrary element, then $f_a := \sum_{i=1}^n \mathrm{ad}(x_i \otimes a) \circ \mathrm{ad}(x^i \otimes a)$ is in $\mathrm{Cent}(\mathfrak{g})$ and $f_a(x \otimes b) = x \otimes ab$ for all $x \in \mathfrak{k}$ and $b \in A$. So $\mathrm{Cent}(\mathfrak{g}) = \mathrm{End}_{\mathfrak{g}}(\mathfrak{g})$ is generated by endomorphisms of the type $\mathrm{ad}(x) \otimes \mathbf{1}$, where $x \in \mathfrak{k}$, and those of the type f_a, where $a \in A$. We obtain:

   $$\mathrm{Cent}(\mathfrak{g}) = \mathrm{End}_{\mathfrak{g}}(\mathfrak{g}) \cong \mathrm{End}_{\mathfrak{k}}(\mathfrak{k}) \otimes A \cong \mathbb{K} \otimes A \cong A.$$

2. If $\mathfrak{g}$ is an arbitary Lie algebra, then the subsets $\mathrm{N}(\mathfrak{g}), \mathrm{S}(\mathfrak{g}) \subseteq \mathrm{Cent}(\mathfrak{g})$ are not necessarily closed under the addition in $\mathrm{End}(\mathfrak{g})$: Let $\mathfrak{g}$ be the abelian complex Lie algebra $\mathbb{C}^2$. Then $\mathrm{Cent}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$ is, as associative algebra, isomorphic to $M_{2,2}(\mathbb{C})$. We can write

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the two matrices on the right hand site are nilpotent, but the matrix on the left hand site is not. And we can write

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the two matrices on the right hand site are semisimple, but the matrix on the left hand site is not.

3. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. The quotient Lie algebra $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ and the Lie subalgebra $\mathfrak{z}(\mathfrak{h}) \subseteq \mathfrak{h}$ are abelian and so the Lie algebra morphisms $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathfrak{z}(\mathfrak{h})$ are just the linear maps $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathfrak{z}(\mathfrak{h})$. In particular, we have $\mathrm{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}], \mathfrak{z}(\mathfrak{h})) = 0$ if and only if $\mathfrak{g}$ is perfect or $\mathfrak{h}$ is centerfree.

   There is a natural isomorphism between linear maps $\overline{\varphi} : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathfrak{z}(\mathfrak{g})$ and linear maps $\varphi : \mathfrak{g} \to \mathfrak{z}(\mathfrak{g})$ with $\mathrm{ad}_x \circ \varphi = 0 = \varphi \circ \mathrm{ad}_x$ for all $x \in \mathfrak{g}$: Given $\overline{\varphi}$, we set $\varphi(x) := \overline{\varphi}(x + [\mathfrak{g},\mathfrak{g}])$. Given φ, the map $\overline{\varphi}$ is well-defined by $\overline{\varphi}(x + [\mathfrak{g},\mathfrak{g}]) := \varphi(x)$ because $x + [\mathfrak{g},\mathfrak{g}] = y + [\mathfrak{g},\mathfrak{g}]$ implies the existence of $a_1, b_1, \dots, a_n, b_n \in \mathfrak{g}$ such that $x - y = \sum_{i=1}^n [a_i, b_i]$ and thus

$$\varphi(x) - \varphi(y) = \sum_{i=1}^n \varphi[a_i, b_i] = \sum_{i=1}^n \varphi \circ \mathrm{ad}_{a_i}(b_i) = 0.$$

   We obtain:

$$\mathrm{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}], \mathfrak{z}(\mathfrak{g})) \cong \{\varphi \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{z}(\mathfrak{g})) \,|\, \mathrm{ad}_x \circ \varphi = 0 = \varphi \circ \mathrm{ad}_x \text{ for all } x \in \mathfrak{g}\} = \mathrm{J}(\mathfrak{g}).$$

4. Let $\mathfrak{g}$ be a Lie algebra such that $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]$. For all $f \in \mathrm{Cent}(\mathfrak{g})$, $g \in \mathrm{J}(\mathfrak{g})$, $x, y \in \mathfrak{g}$ we have:

$$\begin{aligned} fg([x,y]) &= f(0) = 0 \\ [fg(x), y] &= f[g(x), y] = f(0) = 0. \\ gf[x,y] &= g[f(x), y] = 0 \\ [gf(x), y] &= [g(f(x)), y] = 0 \\ g^2(x) &= g(g(x)) \in g(\mathfrak{z}(\mathfrak{g})) \subseteq g([\mathfrak{g},\mathfrak{g}]) = 0. \end{aligned}$$

   Thus, in this case, $\mathrm{J}(\mathfrak{g})$ is a two-sided ideal of $\mathrm{Cent}(\mathfrak{g})$ with $\mathrm{J}(\mathfrak{g})^2 = 0$.

Definition 2.20. A Lie algebra $\mathfrak{g}$ is called *decomposable* if it is the direct sum of two proper ideals. If there is no such decomposition, $\mathfrak{g}$ is called *indecomposable*.

Remark 2.21. A simple Lie algebra is automatically indecomposable, since 0 is its only proper ideal. If an indecomposable Lie algebra $\mathfrak{g}$ is reductive, then it is one-dimensional and trivial or it is simple because $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$ implies $\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$ or $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ and in the latter case a decomposition of the semisimple Lie algebra $[\mathfrak{g},\mathfrak{g}]$ into simple Lie algebras may only have one summand. In general, indecomposable Lie algebras are not reductive, e.g. the Lie algebra $\mathfrak{g}$ in Example 2.15.

Lemma 2.22. *Let $\mathfrak{g}$ be a Lie algebra of finite dimension with abelian centroid, e.g. $\mathfrak{g}$ is perfect or centerfree. Then there exists a decomposition $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ into a direct sum of non-zero ideals $\mathfrak{g}_i$, where each $\mathfrak{g}_i$ is an indecomposable Lie algebra and this decomposition is unique except for the order.*

Proof. The existence of a decomposition into indecomposable non-zero ideals is proven by mathematical induction over the finite dimension of $\mathfrak{g}$. The only difficult part is the one of the uniqueness: Let $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i = \bigoplus_{j=1}^m \mathfrak{h}_j$ be two such decompositions with corresponding systems of orthogonal projections $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$. The projections are in $\mathrm{Cent}(\mathfrak{g})$, as we can show with the following easy calculation:

$$p_i([x, y]) = p_i([x_i, y_i] + [x', y']) = [x_i, y_i] = [x_i + x', y_i] = [x, p_i(y)],$$

where $z = z_i + z'$ is the decompositions of an element $z \in \mathfrak{g}$ into $\mathfrak{g}_i$ and its complement with respect to the decomposition $\bigoplus_{i=1}^n \mathfrak{g}_i$. Since $\mathrm{Cent}(\mathfrak{g})$ is abelian, we have $p_i \circ q_j = q_j \circ p_i$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Now fix an i. Then p_i induces a projection $\mathfrak{h}_j \to \mathfrak{h}_j$ for each $j \in \{1, \ldots, m\}$ and so, by the indecomposability of the $\mathfrak{h}_j$, we obtain $p_i = 0$ on $\mathfrak{h}_j$ or $p_i = \mathbf{1}$ on $\mathfrak{h}_j$. But from this we deduce that $\mathfrak{g}_i = \mathrm{im}(p_i)$ is the direct sum of some $\mathfrak{h}_j$'s. We conclude, by the indecomposability of $\mathfrak{g}_i$, that there is exactly one $j \in \{1, \ldots, m\}$ such that $\mathfrak{g}_i = \mathfrak{h}_j$. □

The following theorem is due to Médina and Revoy, cf. [MeRe93], and gives us information about the centroids of Lie algebras which are not necessarily perfect or centerfree.

Theorem 2.23. *Let $\mathfrak{g}$ be a Lie algebra of finite dimension such that $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$.*

1. *There are indecomposable idempotents $p_1, \ldots, p_n \in \mathrm{Cent}(\mathfrak{g})$ which are pairwise orthogonal, i.e. $p_i \circ p_j = \delta_{ij} p_i$ for $i, j \in \{1, \ldots, n\}$, satisfying $\sum_{i=1}^n p_i = \mathbf{1}$.*
2. *Setting $\mathfrak{g}_i := p_i(\mathfrak{g})$ for $i \in \{1, \ldots, n\}$ we have a decomposition $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ into indecomposable non-zero ideals.*
3. *Each $\mathrm{Cent}(\mathfrak{g}_i)$ is isomorphic to $p_i \circ \mathrm{Cent}(\mathfrak{g}) \circ p_i$, its quotient $\mathrm{Cent}(\mathfrak{g}_i) / \mathrm{J}(\mathfrak{g}_i)$ modulo the maximal nilpotent ideal $\mathrm{J}(\mathfrak{g}_i)$ is a field.*
4. *Setting $C_{ij} := p_i \circ \mathrm{Cent}(\mathfrak{g}) \circ p_j$ for $i, j \in \{1, \ldots, n\}$ we have the decomposition $\mathrm{Cent}(\mathfrak{g}) = \bigoplus_{i,j=1}^n C_{ij}$ into ideals, each C_{ij} is a $\mathrm{Cent}(\mathfrak{g}_i)$-$\mathrm{Cent}(\mathfrak{g}_j)$-bimodule and, if $i \neq j \in \{1, \ldots, n\}$, then the vector space $\mathrm{Hom}(\mathfrak{g}_j / [\mathfrak{g}_j, \mathfrak{g}_j], \mathfrak{z}(\mathfrak{g}_i))$ is isomorphic to C_{ij}. Furthermore, $\mathrm{J}(\mathfrak{g}) = \bigoplus_{i=1}^n \mathrm{J}(\mathfrak{g}_i) \oplus \bigoplus_{i \neq j=1}^n C_{ij}$ is also a decomposition into ideals.*

We also need the following result, Propsition 22.1 of [La01], to show a more general proposition about the uniqueness of the decomposition of a finite-dimensional Lie algebra into indecomposable non-zero ideals.

Theorem 2.24. *Let R be a ring with unity element 1. Suppose there exists a decomposition of $1 \in R$ into a sum of indecomposable idempotents, say $1 = c_1 + \ldots + c_r$, such that $c_i c_j = \delta_{ij} c_i$ and $c_i \in Z(R)$ for all $i, j \in \{1, \ldots, n\}$. Then the decomposition is unique except for the order.*

Proposition 2.25. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra such that $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ with a decomposition into indecomposable non-zero ideals $\bigoplus_{i=1}^n \mathfrak{g}_i$ such that $\mathrm{Hom}(\mathfrak{g}_i / [\mathfrak{g}_i, \mathfrak{g}_i], \mathfrak{z}(\mathfrak{g}_j)) = 0$ if $i \neq j \in \{1, \ldots, r\}$, e.g. $\mathfrak{g}$ is reductive and $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$. Then this decomposition is unique except for the order.*

Proof. By Theorem 2.23, we have $R := \mathrm{Cent}(\mathfrak{g}) \cong \bigoplus_{i=1}^n \mathrm{Cent}(\mathfrak{g}_i)$, $\mathrm{J}(\mathfrak{g}) = \bigoplus_{i=1}^n \mathrm{J}(\mathfrak{g}_i)$ and the quotient $\mathrm{Cent}(\mathfrak{g}) / \mathrm{J}(\mathfrak{g}) \cong \bigoplus_{i=1}^n \mathrm{Cent}(\mathfrak{g}_i) / \mathrm{J}(\mathfrak{g}_i)$ is commutative. Now suppose there are two decompositions of $\mathfrak{g}$ into indecomposable non-zero ideals, say $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i = \bigoplus_{j=1}^m \mathfrak{h}_j$ with corresponding systems of orthogonal projections $p_1, \ldots p_n$ and $q_1, \ldots, q_m$. Let $f \in \mathrm{Cent}(\mathfrak{g})$ and $f = f_1 + \ldots + f_n$ its decomposition into elements of the $\mathrm{Cent}(\mathfrak{g}_i) \subseteq \mathrm{Cent}(\mathfrak{g})$. Then, for fixed $i \in \{1, \ldots, n\}$, we have

$$p_i \circ f = \sum_{k=1}^n p_i \circ f_k = p_i \circ f_i = f_i = f_i \circ p_i = \sum_{k=1}^n f_k \circ p_i = f \circ p_i.$$

So the p_i are in $Z(R)$ and, by a dual argument, all the q_j are so. By Theorem 2.24, this implies $\{p_1, \ldots, p_n\} = \{q_1, \ldots, q_m\}$, thus $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_n\} = \{\mathfrak{h}_1, \ldots, \mathfrak{h}_m\}$. □

Remark 2.26. In general, the decomposition of a finite-dimensional Lie algebra $\mathfrak{g}$ into indecomposable non-zero ideals is not unique and the requirements of the preceding proposition are sharp: Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a decomposition into indecomposable non-zero ideals with corresponding indecomposable projections p_1, p_2 and $\mathrm{Hom}(\mathfrak{g}_1/[\mathfrak{g}_1, \mathfrak{g}_1], \mathfrak{z}(\mathfrak{g}_2)) \neq 0$. If $0 \neq \varphi : \mathfrak{g} \to \mathfrak{g}$ is a morphism mapping $\mathfrak{g}_1$ to $\mathfrak{z}(\mathfrak{g}_2)$ and the spaces $[\mathfrak{g}_1, \mathfrak{g}_1]$ and $\mathfrak{g}_2$ to 0, then $p_1' := (\mathbf{1} + \varphi) \circ p_1 \circ (\mathbf{1} + \varphi)^{-1} = p_1 + \varphi$ is also an indecomposable idempotent of $\mathrm{Cent}(\mathfrak{g})$. Thus $\mathfrak{g} = \mathrm{im}\, p_1' + \mathrm{im}(\mathbf{1} - p_1')$ is another decomposition of $\mathfrak{g}$ into indecomposable non-zero ideals.

Remark 2.27. If $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ are the unique indecomposable non-zero ideals of a Lie algebra $\mathfrak{g}$ and $f \in \mathrm{Aut}(\mathfrak{g})$ is a Lie algebra automorpism, then of course $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ are the unique indecomposable non-zero ideals of $f(\mathfrak{g})$ and thus there is a permutation $\sigma \in S_n$ such that $f(\mathfrak{g}_i) = \mathfrak{g}_{\sigma(i)}$ for all $i \in \{1, \dots, n\}$.

Lemma 2.28. *Let $\mathfrak{g}$ be a Lie algebra of finite dimension.*

1. *If $\mathbb{K} = \mathbb{C}$, then $\mathfrak{g}$ is indecomposable if and only if $\mathrm{S}(\mathfrak{g}) = \mathbb{C} \cdot \mathbf{1}$.*
2. *If $\mathbb{K} = \mathbb{R}$, then $\mathfrak{g}$ is indecomposable if and only if either $\mathrm{S}(\mathfrak{g}) = \mathbb{R} \cdot \mathbf{1}$ or $\mathrm{S}(\mathfrak{g}) = \mathbb{R} \cdot \mathbf{1} + \mathbb{R} \cdot J$ for a complex structure J on $\mathfrak{g}$, i.e $J^2 = -\mathbf{1}$. In the second case, J and $-J$ are the only complex structures on $\mathfrak{g}$ compatible with the Lie bracket.*

Proof.

1. For any $f \in \mathrm{Cent}(\mathfrak{g})$ the eigenspaces of f are ideals of $\mathfrak{g}$: If $y \neq 0$ is an eigenvector of f for the eigenvalue λ and $x \in \mathfrak{g}$ then $f[x, y] = [x, fy] = [x, \lambda y] = \lambda [x, y]$, i.e. $[x, y]$ is also an eigenvector of f for the eigenvalue λ. If $\mathbb{K} = \mathbb{C}$, then the semisimple endomorphisms in $\mathrm{Cent}(\mathfrak{g})$ are exactly the diagonalizable endomorphisms in $\mathrm{Cent}(\mathfrak{g})$. Thus $\mathfrak{g}$ is indecomposable if and only if each endomorphism in $\mathrm{S}(\mathfrak{g})$ has exactly one eigenvalue if and only if $\mathrm{S}(\mathfrak{g}) = \mathbb{C} \cdot \mathbf{1}$.

2. If $\mathbb{K} = \mathbb{R}$ and $\mathfrak{g}$ is decomposable into the direct sum of non-trivial ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with corresponding orthogonal projections p_1 and p_2, then $p_1 p_2 = 0$ although $p_1 \neq 0 \neq p_2$, thus $\mathrm{Cent}(\mathfrak{g})$ has zero-divisors and is neither isomorphic as associative $\mathbb{R}$-algebras to $\mathbb{R}$ nor to $\mathbb{C}$.

   If $\mathbb{K} = \mathbb{R}$ and $\mathfrak{g}$ is indecomposable, then any endomorphism in $\mathrm{S}(\mathfrak{g})$ admits exactly one real eigenvalue or exactly two non-real, complex conjugate eigenvalues. If every endomorphism in $\mathrm{S}(\mathfrak{g})$ admits exactly one real eigenvalue, then $\mathrm{S}(\mathfrak{g}) = \mathbb{R} \cdot \mathbf{1}$. In the other case, if any $f \in \mathrm{S}(\mathfrak{g})$ admits the non-real complex eigenvalues $a + ib$ and $a - ib$, then $J := \frac{1}{b}(f - a\mathbf{1}) \in \mathrm{S}(\mathfrak{g})$ is a complex structure of $\mathfrak{g}$ because of the following calculation for the eigenvector $x \in \mathfrak{g}$ for the eigenvalue $a \pm ib$:

$$\begin{aligned} J^2(x) &= (\frac{1}{b^2}(f - a\mathbf{1})^2)(x) = \frac{1}{b^2}(f^2(x) - 2af(x) + a^2 x) \\ &= \frac{1}{b^2}((a \pm ib)^2 x - 2a(a \pm ib)x + a^2 x) = \frac{1}{b^2}(a^2 \pm 2iab - b^2 - 2a^2 \mp 2iab + a^2)(x) \\ &= -x. \end{aligned}$$

   The space of semisimple endomorphims in $\mathrm{Cent}(\mathfrak{g})$ in this case is the space of real linear combination of $\mathbf{1}$ and J, thus isomorphic to $\mathbb{C}$ as associative algebras over $\mathbb{R}$. Since the complex conjugation is the only $\mathbb{R}$-linear algebra automorphism on $\mathbb{C}$ different from the identity, the only complex structures on $\mathrm{S}(\mathfrak{g})$ are J and $-J$.

□

2.2 Fiber bundles

The definitions of various bundles in this subsection are based on [tD91], [Hu75] and [BiCr01]. Our notion "bundle" is what sometimes is called "fiber bundle", i.e. the fibers of a bundle are all "of the same type".

Definition 2.29.

1. Let E, M, F be manifolds such that there is surjective submersion $\pi : E \to M$ and each fiber $E_x := \pi^{-1}(x)$ for $x \in M$ is diffeomorphic to F. If there is an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of M and a family of smooth maps $\Phi = (\varphi_i)_{i\in I}$, each $\varphi_i : \pi^{-1}(U_i) \to F$ inducing a diffeomorphism $(\pi, \varphi_i) : \pi^{-1}(U_i) \to U_i \times F$ (*local triviality*), then the sextuple $(E, M, \pi, F, \mathfrak{U}, \Phi)$ is called a *bundle* with *bundle space* E over the *base space* M with *projection* π, *fiber* F and *bundle atlas* $(\mathfrak{U}, \Phi)$. A pair (U_i, φ_i) is called *bundle chart*. As a shorter formulation we will say "$\pi : E \to M$ is a bundle with fiber F" without explicit mention of the bundle atlas.

2. A bundle $\pi' : E' \to M'$ is a *subbundle* of a bundle $\pi : E \to M$ provided $E' \subseteq E$ and $M' \subseteq M$ and $\pi'(x) = \pi(x)$ for all $x \in E'$.

Remark 2.30. Since we only consider paracompact manifolds as base spaces, we may always assume that the bundle atlas of a bundle and a corresponding[6] atlas of the base spaces are locally finite.

For each bundle there is a class of interesting functions, the so-called sections.

Definition 2.31. Let $\pi : E \to M$ be a bundle and $k \in \overline{\mathbb{N}}$. A C^k-map

$$\begin{aligned} X : M &\longrightarrow E \\ x &\longmapsto X(x) = X_x \end{aligned}$$

is called C^k-*section* of the bundle, if $\pi \circ X = \mathrm{id}_M$. The set of all C^k-sections of the bundle $\pi : E \to M$ is denoted by $\Gamma^k(E)$ and we also write $\Gamma(E)$ instead of $\Gamma^\infty(E)$.

We now give a first definition of a vector bundle.

Definition 2.32. Let V be a finite-dimensional vector space and $\pi : E \to M$ a bundle with fiber V with a vector space structure on each fiber E_x for $x \in M$, isomorphic to V by $(\varphi_i)_{|E_x} : E_x \to V$ for each $i \in I$. Then the sextuple $(E, M, \pi, V, \mathfrak{U}, \Phi)$ is called a *vector bundle*. As a shorter formulation we will say "$\pi : E \to M$ is a vector bundle with fiber V".

Definition 2.33. Let $\pi_i : E_i \to N$ be a vector bundle with fiber V_i for $i = 1, 2$.

1. A *vector bundle morphism* from $\pi_1 : E_1 \to N$ to $\pi_2 : E_2 \to N$ is a map $\kappa : E_1 \to E_2$ such that the following diagram

$$\begin{tikzcd} E_1 \arrow[r, "\kappa"] \arrow[d, "\pi_1"'] & E_2 \arrow[dl, "\pi_2"] \\ N & \end{tikzcd}$$

   commutes and $\kappa : (E_1)_x \to (E_2)_x$ is a vector space morphism for all $x \in N$. The space of vector bundle morphisms from $\pi_1 : E_1 \to N$ to $\pi_2 : E_2 \to N$ is denoted by $\mathrm{Hom}(E_1, E_2)$. A vector bundle morphism from $\pi_1 : E_1 \to N$ to $\pi_1 : E_1 \to N$ is also called *vector bundle endomorphism* of $\pi_1 : E_1 \to N$.

2. A vector bundle morphism is called *vector bundle map*, if fibers are bijectively mapped on each other and, thus, the fibers are isomorphic vector spaces.

3. A vector bundle morphism κ is a *vector bundle isomorphism*, if there exists a vector bundle morphism $\iota \in \mathrm{Hom}(E_2, E_1)$, such that κ is inverse to ι. In this case the vector bundles $\pi_1 : E_1 \to N$ and $\pi_2 : E_2 \to N$ are called *isomorphic*. Note that vector bundle isomorphisms are vector bundle maps.

[6]The cover of the base space is the same.

A second definition of a vector bundle needs the definition of a principal bundle and a bundle associated to a principal bundle.

Definition 2.34. Let G be a Lie group and $\pi : P \to M$ a bundle with fiber G such that π is identic to the canonical projection of a smooth right action

$$\begin{aligned} R : P \times G &\longrightarrow P \\ (p, g) &\longmapsto p \cdot g = R_g(p) = R^p(g). \end{aligned}$$

If the following diagram commutes for all $i \in I$, $g \in G$

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "R_g"] \arrow[d, "\varphi_i"'] & \pi^{-1}(U_i) \arrow[d, "\varphi_i"] \\ G \arrow[r, "\rho_g"'] & G, \end{tikzcd}$$

then the septuple $(P, M, \pi, G, R, \mathfrak{U}, \Phi)$ is called a *principal bundle* with *structure group* G and *bundle action* R. As a shorter formulation we will say "$\pi : P \to M$ is a G-principal bundle" without explicit mention of the bundle action.

Remark 2.35. Let $\pi : P \to M$ be a G-principal bundle.

1. G acts freely on P by R, i.e. $p \cdot g = p$ for some $p \in P$, $g \in G$ implies $g = 1$. To see this, let $p \cdot g = p$ for some $p \in P$, $g \in G$ and let (U, φ) be a bundle chart in $x := \pi(p)$. We calculate:

$$\varphi(p) = \varphi(p \cdot g) = \varphi(p)g \quad \Longrightarrow \quad 1 = g.$$

   It is easy to show now that R acts simply transitively on its orbits, i.e. for all $p \in P$ the map $R^p : G \to P$ is injective.

2. The condition that the surjective submersion $\pi : P \to M$ is identic to the canonical projection $\pi^R : P \to P/G$ of the action R can be weakened in the following sense: If they both are surjective submersions, then M and P/G are already diffeomorphic. To show this, let h be a map $P/G \to M$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\pi^R"'] & M \\ P/G \arrow[ur, "h"'] & \end{tikzcd} .$$

   The map h is smooth because π^R is a submersion and h is surjective because π is. It is even bijective because $h\left(\pi^R(p)\right) = h\left(\pi^R(q)\right)$ implies $\pi(p) = \pi(q)$ and there exists a bundle chart (U, φ) and an element $g \in G$ such that $(\pi(p), \varphi(p)) = (\pi(q), \varphi(q) \cdot g) = (\pi(q), \varphi(q \cdot g))$ yielding $p = q \cdot g$, thus $\pi^R(p) = \pi^R(q)$. Furthermore, for any $x \in M$ there is an open neighbourhood $U \subseteq M$ such that there exists a section $X : U \to \pi^{-1}(U)$. Then h^{-1} maps $y \in U$ on $\pi^R(X(y)) \in P/G$ and is smooth. So h is a diffeomorphism $P/G \to M$.

If $\pi : P \to M$ is a G-principal bundle and F is a manifold on which G acts from the left, then we can construct a new bundle with fiber F by replacing the structure group G by F.

Definition 2.36. Let $\pi : P \to M$ be a G-principal bundle and F a manifold with a left action

$$\begin{aligned} L : G \times F &\longrightarrow F \\ (g, f) &\longmapsto g \cdot f = L_g(f) = L^f(g). \end{aligned}$$

By defining a left action as follows

$$\begin{aligned}L' : G \times (P \times F) &\longrightarrow P \times F\\ (g,(p,f)) &\longmapsto g \cdot (p,f) := \left(p \cdot g^{-1}, g \cdot f\right).\end{aligned}$$

and considering the orbit space $B := (P \times F)/G$, we obtain a bundle $\pi' : B \to M$ with fiber F, called the *bundle associated* to the G-principal bundle $\pi : P \to M$ with fiber F.[7] Let us examine its structure: The bundle projection is

$$\begin{aligned}\pi' : B &\longrightarrow M\\ [(p,f)] := G \cdot (p,f) &\longmapsto \pi(p)\end{aligned}$$

and this is well-defined because $\pi(p) = \pi(p \cdot g^{-1})$ for all $p \in P, g \in G$. The bundle atlas $(U_i, \varphi_i)_{i \in I}$ of the G-principal bundle $\pi : P \to M$ induces a bundle atlas $(U_i, \varphi_i')_{i \in I}$ of $\pi' : B \to M$ by

$$\begin{aligned}\varphi_i' : \pi'^{-1}(U_i) &\longrightarrow F\\ [(p,f)] &\longmapsto \varphi_i(p) \cdot f\end{aligned}$$

and this is well-defined because $[(p,f)] = [(q,e)]$ yields the existence of $g \in G$ such that $p \cdot g^{-1} = q$ and $g \cdot f = e$ and then $\varphi_i(q) \cdot e = \varphi_i(p \cdot g^{-1}) \cdot g \cdot f = \varphi_i(p) \cdot g^{-1} \cdot g \cdot f = \varphi_i(p) \cdot f$. Finally, each fiber $B_x = \pi'^{-1}(x)$ for $x \in M$ is diffeomorphic to F:

$$\pi'^{-1}(x) = \{\, [(p,f)] \in (P \times F)/G |\, \pi(p) = x\} = \{\, [(p,f)] \in (P \times F)/G |\, p \in P_x\} = (P_x \times F)/G \cong F.$$

Now we can define a vector bundle as being a bundle associated to a principal bundle.

Definition 2.37. Given a finite-dimensional vector space V and a Lie subgroup G of $\mathrm{GL}(V)$ acting on V [8] and a G-principal bundle $\pi : P \to M$, the bundle $\pi' : B \to M$ associated to the G-principal bundle $\pi : P \to M$ with fiber V is called a *vector bundle*.

Remark 2.38.

1. A vector bundle $\pi' : B \to M$ in the sense of Definition 2.37 allows a vector space structure on each fiber B_x for $x \in M$, defined by the condition that all restrictions $(\varphi_i')_{|B_x} : B_x \to V$ are vector space isomorphisms and this is well-defined because

$$\begin{aligned}(\varphi_i')^{-1}_{|B_x}\left(\lambda \varphi_i'(G \cdot (p,v)) + \varphi_i'(G \cdot (q,w))\right) &= (\varphi_i')^{-1}_{|B_x}\left(\lambda \varphi_i(p)(v) + \varphi_i(q)(w)\right)\\ &= (\varphi_i')^{-1}_{|B_x}\left(\varphi_i(p)(\lambda v + w)\right)\\ &= (\varphi_i')^{-1}_{|B_x}\left(\varphi_i'(G \cdot (p, \lambda v + w))\right)\\ &= G \cdot (p, \lambda v + w)\end{aligned}$$

does not depend on the choice of the bundle chart (U_i, φ_i'), where $x \in U_i$, $\lambda \in \mathbb{K}$, $G \cdot (p,v)$, $G \cdot (q,w) \in B_x$. Therefore every vector bundle in the sense of Definition 2.37 is a vector bundle in the sense of Definition 2.32.

2. On the other hand, let $\pi : E \to M$ be a vector bundle with fiber V in the sense of Definition 2.32. By setting

$$P := \dot{\bigcup_{x \in M}} \mathrm{Iso}\,(V, E_x)\,,$$

[7] In this formulation we neglect the dependence of the bundle structure on the actions R and L.

[8] This condition may be replaced by the existence of an smooth group morphism $\Xi : G \to \mathrm{GL}(V)$ (a so-called Lie group representation) and G acting on V as follows: $g.v := \Xi(g)(v)$.

we obtain a surjective submersion $\rho : P \to M$ by setting $\rho(f) := x$ if $f \in \mathrm{Iso}\,(V, E_x)$, where the smooth structure on P is induced by the bundle atlas $(U_i, \varphi_i)_{i\in I}$ of the bundle $\pi : E \to M$ by declaring each map

$$\begin{aligned} \rho^{-1}(U_i) &\longrightarrow U_i \times \mathrm{GL}(V) \\ f &\longmapsto \left(x, (\varphi_i)_{|_{E_{\rho(f)}}} \circ f\right) \end{aligned}$$

to be a diffeomorphism. We obtain a $\mathrm{GL}(V)$-principal bundle $\rho : P \to M$ with bundle action

$$\begin{aligned} P \times \mathrm{GL}(V) &\longrightarrow P \\ (f, T) &\longmapsto f \circ T \end{aligned}$$

and we define $\rho' : B = (P \times V)/\,\mathrm{GL}(V) \to M$ to be the bundle associated to this principal bundle with fiber V. It is isomorphic to the vector bundle $\pi : E \to M$ via

$$\begin{aligned} \kappa : B &\longrightarrow E \\ \mathrm{GL}(V) \cdot (f, v) &\longmapsto f(v). \end{aligned}$$

We will see that, under certain conditions, the structure group may be reduced to a closed subgroup G of $\mathrm{GL}(V)$.

Definition 2.39. Let $\pi' : B \to M$ be a vector bundle with fiber V associated to the G-principal bundle $\pi : P \to M$ with bundle atlas $(U_i, \varphi'_i)_{i\in I}$ and $(U_i, \varphi_i)_{i\in I}$, respectively. We also define an index set as follows: $\mathcal{I} := \{\,(i,j) \in I \times I |\, U_i \cap U_j \neq \emptyset\}$. For $(i,j) \in \mathcal{I}$ and $x \in U_i \cap U_j$ we define an element $g_{ji}(x) \in G$ by

$$\begin{aligned} (\pi, \varphi_j) \circ (\pi, \varphi_i)^{-1} : (U_i \cap U_j) \times G &\longrightarrow (U_i \cap U_j) \times G \\ (x, g) &\longmapsto (x, g_{ji}(x) g)\,. \end{aligned}$$

Note that, if $G \leq GL(V)$ or G is injectively representated in $GL(V)$, this element can also be defined by

$$\begin{aligned} \left(\pi', \varphi'_j\right) \circ \left(\pi', \varphi'_i\right)^{-1} : (U_i \cap U_j) \times V &\longrightarrow (U_i \cap U_j) \times V \\ (x, v) &\longmapsto (x, g_{ji}(x)(v))\,. \end{aligned}$$

We obtain smooth functions $g_{ji} : U_i \cap U_j \to G$ for $(i,j) \in \mathcal{I}$, called *transition maps*. The family $(g_{ji})_{(i,j)\in\mathcal{I}}$ is called *cocyle* relative to the bundle atlas $(U_i, \varphi_i)_{i\in I}$. As a shorter formulation we will say "$\pi' : B \to M$ is a vector bundle with fiber V and cocycle $(g_{ji})_{(i,j)\in\mathcal{I}}$." We have:

$$g_{ji}(x) \cdot g_{ik}(x) \cdot g_{kj}(x) = 1 \text{ for all } x \in U_i \cap U_j \cap U_k. \tag{1}$$

Given another bundle atlas $(U_i, \psi_i)_{i\in I}$ (the cover $\mathfrak{U}$ is the same!) and the cocyle $(k_{ji})_{(i,j)\in\mathcal{I}}$ relative to this bundle atlas, we define smooth mappings $h_i : U_i \to G$ for $i \in I$ by

$$\begin{aligned} (\pi, \psi_i) \circ (\pi, \varphi_i)^{-1} : U_i \times G &\longrightarrow U_i \times G \\ (x, g) &\longmapsto (x, h_i(x) g)\,. \end{aligned}$$

The two cocyles are then linked by the formula

$$k_{ji}(x) = h_j(x) \cdot g_{ji}(x) \cdot h_i(x)^{-1} \text{ for all } x \in U_i \cap U_j. \tag{2}$$

Čech cohomology deals with abstract cocyles fulfilling the conditions (1) and (2) and the equivalence classes of cohomological cocyles. We will see details in Subsection 2.6.

Remark 2.40. If $G \leq \mathrm{GL}(V)$ is a closed subgroup (and hence a Lie subgroup) containing the images of the transition maps of a vector bundle $\pi : E \to M$, then one can show (cf. Theorem I.6.4.1 of [Hu75]) that $\pi : E \to M$ is isomorphic to the bundle associated to a G-principal bundle with base space M and fiber V.

Another equivalent way to define vector bundles is given by Proposition I.5.5.1 of [Hu75].

Proposition 2.41. *Let V be a finite-dimensional vector space, G a closed subgroup of $\mathrm{GL}(V)$, M a manifold, $(U_i)_{i\in I}$ an open cover of M and $g_{ji} : U_i \cap U_j \to G$ for $(i,j) \in \mathcal{I}$ a familiy of maps meeting the condition* (1)*, where $\mathcal{I} := \{(i,j) \in I \times I | U_i \cap U_j \neq \emptyset\}$. Then there is a vector bundle $\pi : E \to M$ with fiber V, bundle atlas $(U_i, \varphi_i)_{i\in I}$ and cocyle $(g_{ji})_{(i,j)\in\mathcal{I}}$ and it is unique up to isomorphism.*

Knowing different ways to define and construct vector bundles, it is now time to give some important examples of vector bundles.

Example 2.42.

1. Let V be a finite-dimensional vector space and M a manifold. Then, by setting $E := M \times V$, we obtain a vector bundle $\pi : E \to M, (m, v) \mapsto m$ with fiber V, called the *trivial* vector bundle with base space M and fiber V.

2. Let M be a m-dimensional smooth manifold. Then the tangent bundle TM is the disjoint union of tangent spaces $\bigcup_{x\in M} T_xM$ provided with a topology and a smooth structure like in [Ne05], Definition II.1.8. If $\pi : TM \to M$, $\pi(T_xM) = \{x\}$ is the natural projection, then $\pi : TM \to M$ is a vector bundle with fiber $\mathbb{R}^m$. The set of smooth sections of this bundle is denoted by $\mathcal{V}(M)$ or $\Gamma(TM)$ and these sections are called *vector fields*.

3. Let $\pi_\ell : E_\ell \to M$ be a vector bundle with fiber V_ℓ, bundle atlas $(U_i, \varphi_i^\ell)_{i\in I}$ and cocyle $(g_{ji}^\ell)_{(i,j)\in\mathcal{I}}$ for $\ell = 1, 2$. Theorem 2.41 yields the existence of the vector bundle $\Pi : \mathrm{Hom}(E_1, E_2) \to M$ with fiber $\mathrm{Hom}(V_1, V_2)$, bundle atlas $(U_i, \Phi_i)_{i\in I}$ and cocyle $(G_{ji})_{(i,j)\in\mathcal{I}}$, where we define the mappings $G_{ji} : U_i \cap U_j \to \mathrm{GL}(\mathrm{Hom}(V_1, V_2))$ by

$$G_{ji}(x)(f) := g_{ji}^2(x) \circ f \circ g_{ji}^1(x)^{-1}$$

for $f \in \mathrm{Hom}(V_1, V_2)$, $x \in U_i \cap U_j$ and $(i,j) \in \mathcal{I}$.

In order to discuss sections of bundles locally, we introduce the following notation.

Definition 2.43. Let (U, φ) be a bundle chart of a vector bundle $\pi : E \to M$ with fiber V. The local form of a section $X \in \Gamma^k(\pi^{-1}(U))$, where $k \in \overline{\mathbb{N}}$, is $X^\varphi \in \mathrm{C}^k(U, V)$, defined by claiming the following diagram to be commutative:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "(\pi,\varphi)"] & U \times V \\ U. \arrow[u, "X"] \arrow[ur, "(\mathrm{id}_U, X^\varphi)"'] & \end{tikzcd}$$

If N is a set and $\eta : \Gamma^k(\pi^{-1}(U)) \to N$ is a map, then the corresponding local form of η is denoted by $\eta^\varphi : \mathrm{C}^k(U, V) \to N$.

A very useful type of cover of the base space of a bundle is the so-called Palais cover.

Definition 2.44. If $\pi : E \to M$ is a bundle, then $\left((V_i, \psi_i, \xi_i, \rho_i)_{i\in J}, (J_t)_{t=1}^r\right)$ is called *Palais cover*, if

(a) $(V_i)_{i\in J}$ is a locally finite cover of M,

(b) $(V_i, \psi_i)_{i\in J}$ is a bundle atlas of E,

(c) $(V_i, \xi_i)_{i\in J}$ is an atlas of M,

(d) $(\rho_i : M \to [0,1])_{i\in J}$ is a partition of unity such that $\mathrm{supp}(\rho_i)$ is a compact subset of V_i for all $i \in J$,

(e) $\{J_1, \ldots, J_r\}$ is a partition of J such that $i, j \in J_t$ and $V_i \cap V_j \neq \emptyset$ already implies $i = j$ for all $t \in \{1, \ldots, r\}$.

If the base space of a bundle is paracompact, then there is always a Palais cover.

Theorem 2.45. *A bundle with paracompact base space possesses a Palais cover.*

Proof. Since M is paracompact, there is surely $(V_j, \psi_j, \xi_j, \rho_j)_{j\in J}$ fulfilling the conditions (a)-(d). By Theorem I in Chapter 1.2 of [GrHaVa72], there is a finite set S and a refinement $(V_{sk})_{s\in S, k\in\mathbb{N}}$ of $(V_j)_{j\in J}$, such that for each $s \in S$ we have $V_{sk} \cap V_{s\ell} = \emptyset$ if $k \neq \ell \in \mathbb{N}$. By restriction of the ψ_j, ξ_j and ρ_j we obtain induced mappings $\psi_{sk}, \xi_{sk}, \rho_{sk}$ and a Palais cover $\Big((V_{sk}, \psi_{sk}, \xi_{sk}, \rho_{sk})_{s\in S, k\in\mathbb{N}}, (\{s0, s1, s2, \ldots\})_{s\in S}\Big)$. $\square$

The following lemma will be very useful in some future proofs.

Lemma 2.46. *If we have a vector bundle $\pi : E \to M$ with fiber V, a section $A \in \Gamma(E)$, a point $x \in M$ and an integer $s \in \mathbb{N}$ such that $j^s_x(A) = 0$, then there is an integer $r \in \mathbb{N}$ and there are sections $A_1, \ldots, A_r \in \Gamma(E)$ and functions $a_1, \ldots, a_r \in \mathrm{C}^\infty(M, \mathbb{R})$ with $a_1(x) = \ldots = a_r(x) = 0$ such that*

$$A = \sum_{i=1}^{r} a_i^{s+1} A_i. \tag{3}$$

Proof. Let n be the vector space dimension of V and m the dimension of M. The lemma will be proven in two steps.

1. We consider the case of M being a convex open neigbourhood $U \subseteq \mathbb{R}^m$ of $x = 0$ and $E = U \times \mathbb{K}^n$ and $A \in \mathrm{C}^\infty(U, \mathbb{K}^n)$ with compact support $F \subseteq U$. Then, by the Taylor Formula, there is a family of functions $A_\alpha \in \mathrm{C}^\infty(U, \mathbb{K}^n)$, $\alpha \in \mathbb{N}^m$, $|\alpha| = s+1$, such that

$$A(y) = \sum_{|\alpha|=s+1} \frac{1}{\alpha!} \cdot y_1^{\alpha_1} \cdots y_m^{\alpha_m} \cdot A_\alpha(y)$$

for all $y \in U$. Let $W \subseteq U$ be an open set with $F \subseteq W$ and $\rho : U \to [0,1]$ be a smooth function with compact support such that $\rho_{|W} \equiv 1$. Then we have

$$A(y) = \rho(y)^{s+2} \cdot A(y) = \sum_{|\alpha|=s+1} (\rho(y)y_1)^{\alpha_1} \cdots (\rho(y)y_m)^{\alpha_m} \cdot \frac{1}{\alpha!} \cdot \rho(y) \cdot A_\alpha(y) \tag{4}$$

for all $y \in U$. By the Multinomial Theorem, we have

$$(t_1 + \ldots + t_m)^k = \sum_{|\alpha|=k} \binom{k}{\alpha!} t_1^{\alpha_1} \cdots t_m^{\alpha_m}$$

for $t_1, \ldots, t_m \in \mathbb{R}$ and, by the inversion formula presented in [MoHeRaCo94], there are, for $\alpha \in \mathbb{N}^m$, scalars $\lambda_{ij} = \lambda_{ij}(\alpha) \in \mathbb{R}$ and $\mu_j = \mu_j(\alpha) \in \mathbb{R}$, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, p\}$ for an integer $p = p(\alpha) \in \mathbb{N}$, such that

$$t_1^{\alpha_1} \cdots t_m^{\alpha_m} = \sum_{j=1}^{p} \mu_j \left(\lambda_{1j} t_1 + \ldots + \lambda_{mj} t_m\right)^{|\alpha|}.$$

Equation (4) then turns into:

$$\begin{aligned} A(y) &= \sum_{|\alpha|=s+1} \sum_{j=1}^{p(\alpha)} \Bigg(\underbrace{\lambda_{1j}\rho(y)y_1 + \ldots + \lambda_{mj}\rho(y)y_m}_{=:a_\alpha(y)} \Bigg)^{s+1} \cdot \underbrace{\frac{\mu_j}{\alpha!} \cdot \rho(y) \cdot A_\alpha(y)}_{=:\widetilde{A}_\alpha(y)} \\ &= \sum_{|\alpha|=s+1} a_\alpha(y)^{s+1} \cdot \widetilde{A}_\alpha(y). \end{aligned}$$

Since $\widetilde{A}_\alpha$ is of compact support and $a_\alpha(0) = 0$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| = s+1$, we have shown (3) for this local case.

2. For the general case, we take an x-neigbourhood $U \subseteq M$, a bundle chart (U, φ), a chart (U, ξ) of M such that $\xi(x) = 0$ and $\xi(U)$ is convex and a smooth function $\rho : M \to [0,1]$ such that $\mathrm{supp}(\rho)$ is a compact subset of U and $\rho_{|W} \equiv 1$ for an x-neighbourhood $W \subseteq U$. By the argument in the first step, we my write

$$(\rho \cdot A)^\varphi \circ \xi^{-1} = \sum_{i=1}^{r} \widetilde{a}_i^{s+1} \widetilde{A}_i$$

for smooth functions $\widetilde{A}_1, \ldots \widetilde{A}_r \in \mathrm{C}^\infty(\xi(U), V)$ and $\widetilde{a}_1, \ldots, \widetilde{a}_r \in \mathrm{C}^\infty(\xi(U), \mathbb{R})$ with $\widetilde{a}_1(x) = \ldots = \widetilde{a}_r(x) = 0$. For $i \in \{1, \ldots r\}$, we define $A_i \in \Gamma(E)$ by $A_i := \rho \cdot A_i'$, where $(A_i')^\varphi(y) := \widetilde{A}_i(\xi(y))$ for all $y \in U$ and we define $a_i \in \mathrm{C}^\infty(M, \mathbb{R})$ with $a_1(x) = \ldots = a_r(x) = 0$ by $a_i := \rho \cdot (\widetilde{a}_i \circ \xi)$. This yields $A = \sum_{i=1}^r a_i^{s+1} A_i$ on M.

□

2.3 Bundles of Lie algebras and the Lie algebra of sections

Now we are ready to define a central objects of this diploma thesis: Bundles of Lie algebras and the corresponding Lie algebra of C^k-sections. The definitions of this subsection are taken from [Le80].

Definition 2.47. Let $\mathfrak{k}$ be a Lie algebra and M a manifold. A *bundle of Lie algebras* $\pi : \mathbb{L} \to M$ with fiber $\mathfrak{k}$ is a vector bundle $\pi : \mathbb{L} \to M$ with fiber $\mathfrak{k}$ which has an $\mathrm{Aut}(\mathfrak{k})$-valued cocyle $(g_{ji})_{(i,j)\in\mathcal{I}}$.

We now define additional structure on a bundle of Lie algebras and on the corresponding space of C^k-sections.

Definition 2.48. Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ and $k \in \overline{\mathbb{N}}$.

1. We justify the name "bundle of Lie algebras" by defining a Lie bracket on each fiber $\mathbb{L}_x$ for $x \in M$ by a bundle chart (U_i, φ_i) in x and the Lie bracket $[\cdot,\cdot]$ on $\mathfrak{k}$ (cf. Example 2.5.2):

$$\begin{aligned} [\cdot,\cdot] : \mathbb{L}_x \times \mathbb{L}_x &\longrightarrow \mathbb{L}_x \\ (v, w) &\longmapsto [v,w]_{\varphi_i} = (\varphi_i)^{-1}_{|\mathbb{L}_x} [\varphi_i(v), \varphi_i(w)]. \end{aligned}$$

In order to see that this is well-defined, we make use of the fact that all maps $g_{ij}(x)$ for $(i,j) \in \mathcal{I}$ are isomorphisms of Lie algebras:

$$\begin{aligned} [v,w]_{\varphi_j} &= (\varphi_j)^{-1}_{|\mathbb{L}_x} [\varphi_j(v), \varphi_j(w)] = (\varphi_i)^{-1}_{|\mathbb{L}_x} (g_{ij}(x) [\varphi_j(v), \varphi_j(w)]) \\ &= (\varphi_i)^{-1}_{|\mathbb{L}_x} [g_{ij}(x)(\varphi_j(v)), g_{ij}(x)(\varphi_j(w))] = (\varphi_i)^{-1}_{|\mathbb{L}_x} [\varphi_i(v), \varphi_i(w)] \\ &= [v,w]_{\varphi_i}. \end{aligned}$$

2. With the help of the Lie algebra structure on each fiber of $\mathbb{L}$, we pointwisely define a Lie bracket on the vector space $\Gamma^k(\mathbb{L})$ by

$$[X, Y](x) := [X(x), Y(x)] \text{ for } X, Y \in \Gamma^k(\mathbb{L}), x \in M.$$

3. $\Gamma^k(\mathbb{L})$ turns out into a *topological* Lie algebra by embedding it into the space $\mathrm{C}^k(M, \mathbb{L})$ equipped with the C^k-compact-open topology, i.e. embed it into the topological product $\prod_{i=0}^k C\left(T^i M, T^i \mathbb{L}\right)$ by $X \mapsto \left(T^i X\right)_{i=0}^k$, where each $C\left(T^i M, T^i \mathbb{L}\right)$ is equipped with the compact-open topology.

Definition 2.49. The Lie algebra $\Gamma^k(\mathbb{L})$ is called *Lie algebra of* C^k*-sections*. In the case of $k = \infty$ we call $\Gamma^\infty(\mathbb{L}) = \Gamma(\mathbb{L})$ also *Lie algebra of smooth sections.*

Remark 2.50.

1. In the case of $M := \{0\}$ we have $\mathbb{L} = \{0\} \times \mathfrak{k} \cong \mathfrak{k}$ and $\Gamma^k(\mathbb{L}) = \mathrm{C}^k(\{0\}, \{0\} \times \mathfrak{k}) \cong \mathfrak{k}$. So the finite-dimensional Lie algebras are special Lie algebras of C^k-sections.

2. The Lie algebra $\Gamma^k(\mathbb{L})$ is a Lie algebra of order zero in the following sense: For sections $X, Y \in \Gamma^k(\mathbb{L})$ and $x \in M$, the expression $[X, Y](x)$ depends on X, Y only via their "zeroth order parts", namely $X(x)$, $Y(x)$. The Lie algebra of smooth vector fields $\mathcal{V}(M) = \Gamma(TM)$ is a Lie algebra of order 1.

There is a useful lemma about Lie algebras of C^k-sections, which we want to prove right now.

Lemma 2.51. *Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ and $k \in \overline{\mathbb{N}}$. Then for all $x \in M$ the evaluation map*

$$\begin{aligned} \mathrm{ev}_x : \Gamma^k(\mathbb{L}) &\longrightarrow \mathbb{L}_x \\ X &\longmapsto X_x \end{aligned}$$

is a surjective morphism of Lie algebras.

Proof. The function ev_x is clearly linear. It is also compatible with the Lie bracket:

$$\mathrm{ev}_x [X, Y] = [X, Y]_x = [X_x, Y_x] = [\mathrm{ev}_x(X), \mathrm{ev}_x(Y)].$$

For the proof of the surjectivity, choose an arbitrary $u \in \mathbb{L}_x$, a bundle chart (U, φ) such that $x \in U$ and let $\rho : U \to \mathbb{R}$ be a smooth map with compact support contained in U and $\rho(x) = 1$. We define $X \in \Gamma(\mathbb{L}) \subseteq \Gamma^k(\mathbb{L})$ via

$$X_y := \begin{cases} \varphi^{-1}_{|\mathbb{L}_x}(\rho(y) \cdot \varphi(u)) & \text{for } y \in U. \\ 0 & \text{for } y \in M \backslash U. \end{cases}$$

Then $\mathrm{ev}_x(X) = X_x = u$. ☐

Remark 2.52. Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ and (U, φ) a bundle chart of $\mathbb{L}$ and (U, ξ) a corresponding chart of M. Any directional derivative ∂_{ξ_i} for $i \in \{1, \ldots, \dim(M)\}$ is a derivation of the poinwise Lie algebra structure on $\mathrm{C}^\infty(U, \mathfrak{k})$, the local forms of the sections in $\Gamma\left(\pi^{-1}(U)\right)$. This follows from the Chain rule, the fact that any bilinear map $\beta : V \times V \to V$ for $\dim(V) < \infty$ is smooth and the rule $T_{(a,b)}\beta(v, w) = \beta(v, b) + \beta(a, w)$. Thus we have the equation:

$$\partial_{\xi_i}[A^\varphi, B^\varphi] = [\partial_{\xi_i} A^\varphi, B^\varphi] + [A^\varphi, \partial_{\xi_i} B^\varphi]$$

for all $A, B \in \Gamma\left(\pi^{-1}(U)\right)$.

There are certain bundles associated to a bundle of Lie algebras which will help us to analyze the structure of a Lie algebra of C^k-sections.

Definition 2.53. Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ associated to an $\mathrm{Aut}(\mathfrak{k})$-principal bundle $\rho : P \to M$.

1. Let $I \trianglelefteq \mathfrak{k}$ be a *characteristic* ideal of $\mathfrak{k}$, i.e. an ideal which is $\mathrm{Aut}(\mathfrak{k})$-stable. We define $\varpi : \mathbb{L}[I] \to M$ to be the bundle associated to $\rho : P \to M$ with fiber I. It can be identified with a subbundle of $\pi : \mathbb{L} \to M$ as follows: A typical element of $\mathbb{L}[I]_x \subseteq \mathbb{L}[I]$ takes the form
$$[(\psi, i)] := \left\{(\psi f^{-1}, f(i)) : f \in \mathrm{Aut}(\mathfrak{k})\right\}$$
for a Lie algebra isomorphism $\psi : \mathfrak{k} \to \mathbb{L}_x$ and an element $i \in I$. The element $\psi(i) \in \mathbb{L}_x$ is clearly well-defined and so we may embed $\mathbb{L}[I]$ into $\mathbb{L}$. In the cases of I being $[\mathfrak{k}, \mathfrak{k}]$ and $\mathfrak{z}(\mathfrak{k})$ [9] we also write $[\mathbb{L}, \mathbb{L}]$ instead of $\mathbb{L}[[\mathfrak{k}, \mathfrak{k}]]$, called the *commutator* of $\mathbb{L}$ and $\mathfrak{z}(\mathbb{L})$ instead of $\mathbb{L}[\mathfrak{z}(\mathfrak{k})]$, called the *center* of $\mathbb{L}$. Note that $[\mathbb{L}, \mathbb{L}]_x \cong [\mathbb{L}_x, \mathbb{L}_x]$ and $\mathfrak{z}(\mathbb{L})_x \cong \mathfrak{z}(\mathbb{L}_x)$ via $[(\psi, i)] \mapsto \psi(i)$ for $i \in [\mathfrak{k}, \mathfrak{k}]$ and $i \in \mathfrak{z}(\mathfrak{k})$, respectively.

2. Let $J \subseteq \mathrm{End}(\mathfrak{k})$ be a subalgebra which is invariant under the left action
$$\begin{aligned} \mathrm{Aut}(\mathfrak{k}) \times \mathrm{End}(\mathfrak{k}) &\longrightarrow \mathrm{End}(\mathfrak{k}) \\ (f, g) &\longmapsto f \circ g \circ f^{-1}. \end{aligned}$$
We define $\varpi : \mathbb{L}(J) \to M$ to be the bundle associated to $\rho : P \to M$ with fiber J. It can be identified with a subbundle of $\Pi : \mathrm{Hom}(\mathbb{L}, \mathbb{L}) \to M$ as follows: A typical element of $\mathbb{L}(J)_x \subseteq \mathbb{L}(J)$ takes the form
$$[(\psi, j)] := \mathrm{Aut}(\mathfrak{k}) \cdot (\psi, j) = \left\{(\psi f^{-1}, f j f^{-1}) : f \in \mathrm{Aut}(\mathfrak{k})\right\}$$
for a Lie algebra isomorphism $\psi : \mathfrak{k} \to \mathbb{L}_x$ and an endomorphism $j : \mathfrak{k} \to \mathfrak{k}$, $j \in J$. The map $\psi \circ j \circ \psi^{-1} : \mathbb{L}_x \longrightarrow \mathbb{L}_x$ is clearly well-defined and, by applying this construction to all classes $[(\psi, j)]$ for $\psi \in \mathrm{Iso}(\mathfrak{k}, \mathbb{L}_x)$, where x runs through M and $j \in J$ is fixed, we obatin an element of $\mathrm{Hom}(\mathbb{L}, \mathbb{L})$. In the cases of J being $\mathrm{Der}(\mathfrak{k})$ and $\mathrm{Cent}(\mathfrak{k})$ [10] we also write $\mathrm{Der}\,(\mathbb{L})$ instead of $\mathbb{L}(\mathrm{Der}(\mathfrak{k}))$ and $\mathrm{Cent}(\mathbb{L})$ instead of $\mathbb{L}(\mathrm{Cent}(\mathfrak{k}))$. Note that $\mathrm{Der}(\mathbb{L})_x \cong \mathrm{Der}(\mathbb{L}_x)$ and $\mathrm{Cent}(\mathbb{L})_x \cong \mathrm{Cent}(\mathbb{L}_x)$ via $[(\psi, j)] \mapsto \psi \circ j \circ \psi^{-1}$ for $j \in \mathrm{Der}(\mathfrak{k})$ and $j \in \mathrm{Cent}(\mathfrak{k})$, respectively.

2.4 Lie connections

We briefly repeat the definition of a covariant derivative and define the Lie connection corresponding to a fixed bundle of Lie algebras.

Definition 2.54. Let $\pi : E \to M$ be a vector bundle with fiber V. A *covariant derivative* or *connection* of E is a linear map
$$\begin{aligned} \nabla : \Gamma(TM) &\longrightarrow \mathrm{End}(\Gamma(E)) \\ X &\longmapsto \nabla_X \end{aligned}$$
satisfying the following conditions:

---

[9] These ideals are characteristic:

(a) $i = \sum_j [x_j, y_j] \implies f(i) = \sum_j [f x_j, f y_j]$ for all $f \in \mathrm{Aut}(\mathfrak{k})$, $i \in [\mathfrak{k}, \mathfrak{k}]$, $x_j, y_j \in \mathfrak{k}$.

(b) $\left[f^{-1}x, i\right] = 0 \implies f\left[f^{-1}x, i\right] = 0 \implies [x, f(i)] = 0$ for all $f \in \mathrm{Aut}(\mathfrak{k})$, $i \in \mathfrak{z}(\mathfrak{k})$, $x \in \mathfrak{k}$.

[10] These algebras are $\mathrm{Aut}(\mathfrak{k})$-invariant:

(a) $fgf^{-1}[x, y] = fg(\left[f^{-1}x, f^{-1}y\right]) = f(\left[gf^{-1}x, f^{-1}y\right] + \left[f^{-1}x, gf^{-1}y\right]) = \left[fgf^{-1}x, y\right] + \left[x, fgf^{-1}y\right]$ for all $f \in \mathrm{Aut}(\mathfrak{k})$, $g \in \mathrm{Der}(\mathfrak{k})$, $x, y \in \mathfrak{k}$.

(b) $(fgf^{-1} \circ \mathrm{ad}_x)(y) = fgf^{-1}[x, y] = fg(\left[f^{-1}x, f^{-1}y\right]) = (fg \circ \mathrm{ad}_{f^{-1}x})(f^{-1}y) = (f \circ \mathrm{ad}_{f^{-1}x})(gf^{-1}y) = f\left[f^{-1}x, gf^{-1}y\right] = \left[x, fgf^{-1}y\right] = (\mathrm{ad}_x \circ fgf^{-1})(y)$ for all $f \in \mathrm{Aut}(\mathfrak{k})$, $g \in \mathrm{Cent}(\mathfrak{k})$, $x, y \in \mathfrak{k}$.

(a) $\nabla_X(aA) = (X.a)A + a\nabla_X A$ for all $X \in \Gamma(TM), a \in \mathrm{C}^\infty(M,\mathbb{K}), A \in \Gamma(E)$.

(b) $\nabla_{(aX+bY)}A = a\nabla_X A + b\nabla_Y A$ for all $X, Y \in \Gamma(TM), a, b \in \mathrm{C}^\infty(M,\mathbb{K}), A \in \Gamma(E)$.

Remark 2.55. Given a point $x \in M$, a vector field $X \in \Gamma(TM)$ and sections $A, B \in \Gamma(E)$ it is, by using property (b) of Definition 2.54, easy to see that $((\nabla_X A)B)(x)$ depends on X only via X_x:

Firstly we show that $((\nabla_X A)B)(x)$ depends on X at most via $X_{|U}$, where $U \subseteq M$ is any open neighbourhood of x. In fact, if $\rho : M \to [0,1]$ is a smooth map with $\rho_{|M\setminus U} \equiv 0$ and $\rho_{|W} \equiv 1$ for a smaller x-neighbourhood $W \subseteq U$, then $(X - X')_{|U} \equiv 0$ implies

$$(\nabla_X Y)(x) - (\nabla_{X'} Y)(x) = (\nabla_{X-X'} Y)(x) = \rho(x) \cdot (\nabla_{X-X'} Y)(x) = (\nabla_{\rho\cdot(X-X')} Y)(x) = (\nabla_0 Y)(x) = 0.$$

Thus we can now reduce the problem to a problem in a chart (U, ξ): Let X, X' be two vector fields identical in x and $\rho : M \to [0,1]$ as above. We can write $X - X' = \sum_{i=1}^m Z^i \partial_{\xi_i}$ for certain functions $Z^1, \dots Z^m \in \mathrm{C}^\infty(U,\mathbb{R})$. We note that $Z^i(x) = 0$ for all i and conclude by calculating:

$$\begin{aligned}(\nabla_X Y)(x) - (\nabla_{X'} Y)(x) &= (\nabla_{\rho\cdot(X-X')} Y)(x) = \left(\nabla_{\rho\cdot\left(\sum_{i=1}^m Z^i\partial_{\xi_i}\right)} Y\right)(x) \\ &= \left(\sum_{i=1}^m \left(\rho \cdot Z^i\right) \cdot \nabla_{\partial_{\xi_i}} Y\right)(x) = \sum_{i=1}^m Z^i(x) \cdot \left(\nabla_{\partial_{\xi_i}} Y\right)(x) = 0.\end{aligned}$$

Locally, a covariant derivative ist given by the so-called Christoffel symbols.

Definition 2.56. Let (U, φ) be a bundle chart of the vector bundle $\pi : E \to M$ with fiber V and (U, ξ) a corresponding chart of M and let ∇ be a covariant derivative of E. Then there are unique mappings $\Gamma_i^\varphi \in \mathrm{C}^\infty(U, \mathrm{End}(V))$ for $i \in \{1, \dots, \dim(M)\}$ such that

$$(\nabla_X A)^\varphi = \sum_{i=1}^{\dim(M)} X^i \left(\partial_{\xi_i} A^\varphi + \Gamma_i^\varphi \cdot A^\varphi\right)$$

for all sections A and all vector fields X with local form $\sum_{i=1}^m X^i \partial_{\xi_i}$. The maps Γ_i^φ are called *Christoffel symbols* of ∇ with respect to (U, φ).

For a given bundle of Lie algebras we now define a Lie connection.

Definition 2.57. Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ and ∇ a covariant derivative of $\mathbb{L}$. We call ∇ a *Lie connection* if for all vector fields $X \in \Gamma(TM)$ and for all sections $A, B \in \Gamma(\mathbb{L})$ we have

$$\nabla_X [A, B] = [\nabla_X A, B] + [A, \nabla_X B],$$

i.e. ∇ is a map $\Gamma(TM) \to \mathrm{Der}(\Gamma(\mathbb{L}))$.

Whether a covariant derivative of a given bundle of Lie algebras is a Lie connection, can be decided by examining its Christoffel symbols.

Lemma 2.58. *Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ and ∇ a covariant derivative of $\mathbb{L}$. Then ∇ is a Lie connection if and only if for all bundle charts (U, φ) of $\mathbb{L}$ and all corresponding charts (U, ξ) of M the Christoffel symbols are in in $\mathrm{C}^\infty(U, \mathrm{Der}(\mathfrak{k}))$.*

Proof. The covariant derivative ∇ is a Lie connection if and only if for all bundle charts (U, φ) of $\mathbb{L}$, all corresponding charts (U, ξ) of M, all sections $A, B \in \Gamma\left(\pi^{-1}(U)\right)$ and any integer $i \in \{1, \dots, \dim(M)\}$ we have

$$\partial_{\xi_i}[A^\varphi, B^\varphi] + \Gamma_i^\varphi [A^\varphi, B^\varphi] = [\partial_{\xi_i} A^\varphi + \Gamma_i^\varphi(A^\varphi), B^\varphi] + [A^\varphi, \partial_{\xi_i} B^\varphi + \Gamma_i^\varphi(B^\varphi)]. \tag{5}$$

Due to Remark 2.52 we have

$$\partial_{\xi_i}\left[A^\varphi, B^\varphi\right] = \left[\partial_{\xi_i} A^\varphi, B^\varphi\right] + \left[A^\varphi, \partial_{\xi_i} B^\varphi\right].$$

So the equation (5) turns into

$$\Gamma_i^\varphi(x)\left[A^\varphi(x), B^\varphi(x)\right] = \left[\Gamma_i^\varphi(x)\left(A^\varphi(x)\right), B^\varphi(x)\right] + \left[A^\varphi(x), \Gamma_i^\varphi(x)\left(B^\varphi(x)\right)\right]$$

for all $x \in U$. But the last equation is, due to Lemma 2.51, equivalent to $\Gamma_i^\varphi \in \mathrm{C}^\infty(U, \mathrm{Der}(\mathfrak{k}))$. □

Finally we want to show the existence of Lie connections.

Proposition 2.59. *Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$. Then there exists a covariant derivative ∇ of $\mathbb{L}$.*

Proof. Let $\pi : \mathbb{L} \to M$ be a bundle of Lie algebras with fiber $\mathfrak{k}$ and $(U_j, \varphi_j)_{j\in J}$ a locally finite bundle atlas of $\mathbb{L}$ with corresponding atlas $(U_j, \xi_j)_{j\in J}$ of M. For each $j \in J$ we define a covariant derivative ∇^{U_j} on $\pi^{-1}(U_j)$ by:

$$\left(\nabla_X^{U_j} A\right)^{\varphi_j} := \sum_{i=1}^{\dim(M)} X^i \partial_{(\xi_j)_i} A^\varphi$$

for $X \in \Gamma(TU)$ and $A \in \Gamma\left(\pi^{-1}(U_j)\right)$. This covariant derivative is a Lie connection (cf. Lemma 2.58). After choosing a smooth partition $\left(\rho_j : M \to [0,1]\right)_{j\in J}$ of 1 subordinated to the cover $(U_j)_{j\in J}$, i.e. $\mathrm{supp}(\rho_j)$ is a compact subset of U_j for all $j \in J$, we define a covariant derivative ∇ of $\mathbb{L}$ by

$$\nabla := \sum_{j\in J} \rho_j \cdot \nabla^{U_j}.$$

Since ∇ is locally the finite sum of Lie connections, it is also a Lie connection. □

2.5 Local operators and the Peetre Theorems

Definition 2.60. Let $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ be two vector bundles and $k_1, k_2 \in \overline{\mathbb{N}}$. A linear operator $T : \Gamma^{k_1}(E_1) \to \Gamma^{k_2}(E_2)$ is called *local*, if for any $X \in \Gamma^{k_1}(E_1)$ and any open set $U \subseteq M$ the condition $X_x = 0 \in (E_1)_x$ for all $x \in U$ implies $(TX)_x = 0 \in (E_2)_x$ for all $x \in U$.

The meaning of an operator to be local is shown in the following lemma.

Lemma 2.61. *If $Y, Z \in \Gamma^{k_1}(E_1)$ are identical sections on an open set $U \subseteq M$ and we have a local operator $T : \Gamma^{k_1}(E_1) \to \Gamma^{k_2}(E_2)$, then $TY, TZ \in \Gamma^{k_1}(E_2)$ are identical on U.*

Proof. The section $X := Y - Z \in \Gamma^{k_1}(E_1)$ vanishes on U. The locality of T yields that $TX = TY - TZ$ also vanishes on U. So $TY_{|U} = TZ_{|U}$. □

Example 2.62. Let $\nabla : \Gamma(TM) \longrightarrow \mathrm{End}(\Gamma(E))$ be a covariant derivative on a vector bundle $\pi : E \to M$ with fiber V and $X \in \Gamma(TM)$ a vector field. Then ∇_X is a local operator:

Let $Y \in \Gamma(E)$ be a section zero on an open set $U \subseteq M$, let $x \in U$ be a point and let $\rho : M \to [0,1]$ be a smooth map such that $\rho_{|M\setminus U} \equiv 0$ and $\rho_{|W} \equiv 1$ for a smaller x-neighbourhood $W \subseteq U$. We easily calculate:

$$(\nabla_X Y)(x) = \rho(x)(\nabla_X Y)(x) = (\nabla_X(\rho \cdot Y))(x) - ((X.\rho)Y)(x) = 0 - 0 = 0.$$

The Peetre Theorem, proven in [Na68], p. 175-176, can be stated as follows:

Theorem 2.63. (Peetre Theorem, Version 1) *Let M be a smooth m-dimensional manifold and $\pi_i : E_i \to M$ a vector bundle with fiber V_i for $i = 1, 2$. If $T : \Gamma(E_1) \to \Gamma(E_2)$ is a local operator, then for any point $x \in M$ there exists an open neighbourhood $U \subseteq M$, bundle charts (U, φ_1), (U, φ_2) of E_1, E_2, respectively, a chart (U, ξ) of M, a number $n \in \mathbb{N}$ and a familiy of functions $f_\alpha \in \mathrm{C}^\infty(U, \mathrm{Hom}(V_1, V_2))$, $\alpha \in \mathbb{N}^m$, $|\alpha| \leq n$, such that for all $X \in \Gamma\left(\pi_1^{-1}(U)\right)$ we have*

$$(TX)^{\varphi_2} = \sum_{|\alpha| \leq n} f_\alpha \cdot \left(\partial_\xi^\alpha X^{\varphi_1}\right). \tag{6}$$

Definition 2.64. The formula (6) says that T is a *differential operator of order at most n on U*.

By a modification of the proof of Theorem 2.63 one gets the following result (cf. [Le79], p.52):

Theorem 2.65. (Peetre Theorem, Version 2) *Let M be a smooth m-dimensional manifold and $\pi_i : E_i \to M$ a vector bundle with fiber V_i and $k_i \in \mathbb{N}$ for $i = 1, 2$. If $T : \Gamma^{k_1}(E_1) \to \Gamma^{k_2}(E_2)$ is a local operator, then T is a differential operator of order at most $k_1 - k_2$. In particular, if $k_1 = k_2$ then T is a differential operator of order 0 and if $k_1 < k_2$ then $T = 0$. Furthermore, we obtain $T = 0$ if $T : \Gamma^{k_1}(E_1) \to \Gamma(E_2) \subseteq \Gamma^{k_1+1}(E_2)$ is local for $k_1 \in \mathbb{N}$.*

Remark 2.66. In the situations of the above theorems, if $T : \Gamma^k(E_1) \to \Gamma^k(E_2)$ is a differential operator of order 0 for $k \in \overline{\mathbb{N}}$, then it can be identified with a C^k-section of the bundle $\mathrm{Hom}(E_1, E_2)$ as follows: Fix an $x \in M$ and an open neighbourhood $U \subseteq M$ of x such that $\pi_1^{-1}(U)$ is trivial. For vectors $v \in (E_1)_x$ we choose sections $X : U \to \pi_1^{-1}(U)$ such that $X_x = v$ and define the linear map

$$\begin{aligned} \tau_x : (E_1)_x &\longrightarrow (E_2)_x \\ v &\longmapsto (TX)_x . \end{aligned}$$

depends on X only via X_x because T is an operator of order 0. The required section is the map $M \to \mathrm{Hom}(E_1, E_2)$, $x \mapsto \tau_x$.

2.6 Čech cohomology

Čech cohomology is one of the important cohomology theories in algebraic topology. It can be defined for any topological space and a presheaf of groups on this space. However, we will only consider it for a manifold M and a discrete group G, which can be understood as a Lie group. The following definitions are due to [tD91].

Definition 2.67. Let $\mathfrak{Cov}(M) := \{\mathfrak{V} \in \mathfrak{P}(\mathcal{O}(M)) | \bigcup\{U \in \mathfrak{V}\} = M\}$ denote the set of collections of open subsets of M who cover M and for $\mathfrak{V} \in \mathfrak{Cov}(M)$ we define $\mathfrak{V} * \mathfrak{V} := \{(U, V) \in \mathfrak{V} \times \mathfrak{V} | U \cap V \neq \emptyset\}$.

1. A familiy of smooth functions $(g_{VU} : U \cap V \to G)_{(U,V) \in \mathfrak{V}*\mathfrak{V}}$ is called a *cocyle* related to $\mathfrak{V} \in \mathfrak{Cov}(M)$, if it satisfies the equation

   $$g_{VU}(x) \cdot g_{UW}(x) \cdot g_{WV}(x) = 1$$

   for all $x \in U \cap V \cap W$ for all $U, V, W \in \mathfrak{V}$. The set of these families is denoted by $\check{\mathrm{Z}}^1(M, \mathfrak{V}, G)$.

2. Two cocycles $(g_{VU})_{(U,V) \in \mathfrak{V}*\mathfrak{V}}, (k_{VU})_{(U,V) \in \mathfrak{V}*\mathfrak{V}} \in \check{\mathrm{Z}}^1(M, \mathfrak{V}, G)$ are called *cohomologous*, if there exists a familiy of smooth mappings $(h_U : U \to G)_{U \in \mathfrak{V}}$ such that the relation

   $$k_{VU}(x) = h_V(x) \cdot g_{VU}(x) \cdot (h_U(x))^{-1}$$

   is satisfied for all $x \in U \cap V$ for all $U, V \in \mathfrak{V}$. The relation "cohomologous" is an equivalence relation on $\check{\mathrm{Z}}^1(M, \mathfrak{V}, G)$ and the set of the corresponding equivalence classes is denoted by $\check{\mathrm{H}}^1(M, \mathfrak{V}, G)$.

3. If G is abelian, two cocycles $(g_{VU})_{(U,V)\in\mathfrak{V}*\mathfrak{V}}, (k_{VU})_{(U,V)\in\mathfrak{V}*\mathfrak{V}} \in \check{\mathrm{Z}}^1(\mathfrak{V},G)$ can be multiplied pointwisely and we get a group structure on $\check{\mathrm{Z}}^1(M,\mathfrak{V},G)$. The set of cocyles $x \mapsto h_V(x)\cdot(h_U(x))^{-1}$ is a normal subgroup of $\check{\mathrm{Z}}^1(M,\mathfrak{V},G)$, denoted by $\check{\mathrm{B}}^1(M,\mathfrak{V},G)$ and this gives rise to the group structure on the quotient $\check{\mathrm{Z}}^1(M,\mathfrak{V},G)/\,\check{\mathrm{B}}^1(M,\mathfrak{V},G) = \check{\mathrm{H}}^1(M,\mathfrak{V},G)$.
4. $\mathfrak{Cov}(M)$ is a directed set by the refinement relation

$$\mathfrak{V} \prec \mathfrak{W} \quad :\Longleftrightarrow \quad \forall W \in \mathfrak{W}\, \exists V \in \mathfrak{V} :\ W \subseteq V$$

and so we can define the *first Čech cohomology group* of M with values in G to be[11]

$$\check{\mathrm{H}}^1(M,G) := \varinjlim_{\mathfrak{V}\in\mathfrak{Cov}(M)} \check{\mathrm{H}}^1(M,\mathfrak{V},G).$$

In general, if G is not necessarily abelian, we obtain the *first Čech cohomology set* $\check{\mathrm{H}}^1(M,G)$ without group structure.

If G is abelian and discrete, then the Čech cohomology groups $\check{\mathrm{H}}^1(M,G)$ are isomorphic to the singular cohomology groups $\mathrm{H}^1(M,G)$ (cf. e.g [EiSt57], Chapter IX), which can be easily "calculated" in some cases. There are also group isomorphies $\check{\mathrm{H}}^1(M,G) \cong \mathrm{Hom}(\mathrm{H}_1(M),G) \cong \mathrm{Hom}(\pi_1(M),G)$, where $\mathrm{H}_1(M)$ is the first singular homology group of M with values in $\mathbb{Z}$ and $\pi_1(M)$ is the fundamental group of M.

Remark 2.68. We give some first cohomology groups which are calculated in any elementary algebraic topology lecture:

1. If G is an abelian group and M is a simply connected manifold, then $\mathrm{H}^1(M,G) = 0$.
2. If G is an abelian group, then $\mathrm{H}^1(\mathbb{S}^n,G) = \begin{cases} G & \text{if } n=1 \\ 0 & \text{if } n>1. \end{cases}$
3. $\mathrm{H}^1(\mathbb{C}P^n,\mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^1(\mathbb{H}P^n,\mathbb{Z}/2\mathbb{Z}) = 0$.

2.7 Structure of Lie algebras of C^k-sections

From now on, facts from [Le80] are explained in detail and generalized. We have $k \in \overline{\mathbb{N}}$ and a bundle of Lie algebras $\pi : \mathbb{L} \to M$ with fiber $\mathfrak{k}$. The dimensions of M and $\mathfrak{k}$ are denoted by m and d, respectively. Lecomte only considered the case of $\mathfrak{z}(\mathfrak{k}) = 0$ and we will replace this condition with weaker ones.

2.7.1 Center and commutator

The center of $\Gamma^k(\mathbb{L})$ is easy to calculate.

Proposition 2.69. *For $k \in \overline{\mathbb{N}}$ we have $\mathfrak{z}\left(\Gamma^k(\mathbb{L})\right) = \Gamma^k\left(\mathfrak{z}(\mathbb{L})\right)$.*

Proof. If $X \in \Gamma^k\left(\mathfrak{z}(\mathbb{L})\right)$ then, for all $x \in M$, we have $X_x \in \mathfrak{z}(\mathbb{L}_x)$ and thus $[X,Y]_x = [X_x,Y_x] = 0$ for all $Y \in \Gamma^k(\mathbb{L})$ and $x \in M$. This proves $\mathfrak{z}\left(\Gamma^k(\mathbb{L})\right) \supseteq \Gamma^k\left(\mathfrak{z}(\mathbb{L})\right)$. If $X \in \mathfrak{z}\left(\Gamma^k(\mathbb{L})\right)$, $x \in M$ and $u \in \mathbb{L}_x$, then, by Lemma 2.51, there is a section $Y \in \Gamma^k(\mathbb{L})$ such that $Y_x = u$, thus $[X_x,u] = [X,Y]_x = 0$ proving $X_x \in \mathfrak{z}(\mathbb{L}_x)$. This proves $\mathfrak{z}\left(\Gamma^k(\mathbb{L})\right) \subseteq \Gamma^k\left(\mathfrak{z}(\mathbb{L})\right)$. ☐

Corollary 2.70. $\Gamma^k(\mathbb{L})$ *is centerfree if and only if $\mathfrak{k}$ is centerfree.*

[11] This definition is not rigorous. If $(W_j)_{j\in J}$ is a refinement of $(V_i)_{i\in I}$, then, for the definition of the direct limit, it is necessary to have a function $f : J \to I$ such that $f(j) = i$ implies $W_j \subseteq V_i$. But one can show that $\check{\mathrm{H}}^1(M,G)$ does not depend on the choice of such functions.

For the calculation of the commutator of $\Gamma^k(\mathbb{L})$ we first have to prove a technical lemma.

Lemma 2.71. *There exists an $r \in \mathbb{N}$ such that, for each bundle chart (U, φ) and each $X \in \Gamma^k([\mathbb{L}, \mathbb{L}])$ with compact support contained in U, there are sections $Y_1, Z_1, \ldots, Y_r, Z_r \in \Gamma^k(\mathbb{L})$ with compact supports contained in U with*

$$X = \sum_{t=1}^{r} [Y_t, Z_t] .$$

Proof. Let $([u_1, w_1], \ldots [u_r, w_r])$ be a basis of $[\mathfrak{k}, \mathfrak{k}]$. Let $F \subseteq M$ be compact such that

$$\operatorname{supp} X \subseteq F^\circ \subseteq F \subseteq U$$

and $\rho : M \to [0, 1]$ a smooth function with compact support contained in U such that $\rho_{|F} \equiv 1$. Then there are functions $f_1, \ldots, f_r \in \mathrm{C}^k(M, \mathbb{K})$ with supports contained in F such that we have, for each $x \in U$:

$$X^\varphi(x) = \sum_{t=1}^{r} f_t(x) \, [u_t, w_t] .$$

For $t \in \{1, \ldots, r\}$ we define $Y_t, Z_t \in \Gamma^k(\mathbb{L})$ by

$$Y_t(x) = Z_t(x) = 0 \text{ for } x \in M \backslash U$$
$$Y_t^\varphi(x) = f_t(x) u_t \text{ and } Z_t^\varphi(x) = \rho(x) w_t \text{ for } x \in U.$$

For $x \in M \backslash F$ we have $X(x) = 0 = \sum_{t=1}^{r} 0 = \sum_{t=1}^{r} [Y_t(x), Z_t(x)]$. For $x \in F$ we have

$$X^\varphi(x) = \sum_{t=1}^{r} [f_t(x) u_t, 1 \cdot w_t] = \sum_{t=1}^{r} [Y_t^\varphi(x), Z_t^\varphi(x)] .$$

Thus $X = \sum_{t=1}^{r} [Y_t, Z_t]$. ◻

Proposition 2.72. *For $k \in \overline{\mathbb{N}}$ we have $\left[\Gamma^k(\mathbb{L}), \Gamma^k(\mathbb{L})\right] = \Gamma^k([\mathbb{L}, \mathbb{L}])$.*

Proof. Let $\left((V_i, \psi_i, \xi_i, \rho_i)_{i \in J}, (J_t)_{t=1}^r\right)$ be a Palais cover (cf. Definition 2.44 and Theorem 2.45). Any section $X \in \Gamma^k(\mathbb{L})$ takes the form $X = \sum_{t=1}^{r} X_t$ for $X_t = \sum_{i \in J_t} \rho_i X$, where $t \in \{1, \ldots, r\}$. We fix $X \in \Gamma^k([\mathbb{L}, \mathbb{L}])$ and $t \in \{1, \ldots, r\}$. Then, by Lemma 2.71, there are, for each $i \in J_t$, sections $Y_1^i, Z_1^i, \ldots, Y_s^i, Z_s^i \in \Gamma^k(\mathbb{L})$ with compact supports contained in V_i, such that $\rho_i X = \sum_{p=1}^{s} \left[Y_p^i, Z_p^i\right]$. By setting $Y_p := \sum_{i \in J_t} Y_p^i \in \Gamma^k(\mathbb{L})$ and $Z_p := \sum_{i \in J_t} Z_p^i \in \Gamma^k(\mathbb{L})$ for each $p \in \{1, \ldots, s\}$ and recalling that the supports of Y_p^i, Z_p^j for $i \neq j$ are disjoint, we obtain $X_t = \sum_{p=1}^{s} [Y_p, Z_p]$, yielding $X = \sum_{t=1}^{r} X_t$ and $\left[\Gamma^k(\mathbb{L}), \Gamma^k(\mathbb{L})\right] \supseteq \Gamma^k([\mathbb{L}, \mathbb{L}])$.

In order to show $\left[\Gamma^k(\mathbb{L}), \Gamma^k(\mathbb{L})\right] \subseteq \Gamma^k([\mathbb{L}, \mathbb{L}])$ we consider $X = \sum_{s=1}^{p} [Y_p, Z_p]$ for appropriate sections $Y_p, Z_p \in \Gamma^k(\mathbb{L})$. For arbitary $x \in M$ we have $X(x) = \sum_{s=1}^{p} [Y_p(x), Z_p(x)] \in [\mathbb{L}_x, \mathbb{L}_x] = [\mathbb{L}, \mathbb{L}]_x$. ◻

Corollary 2.73. $\Gamma^k(\mathbb{L})$ *is perfect if and only if* $\mathfrak{k}$ *is perfect.*

2.7.2 Derivations

Now we want to calculate the derivations of the Lie algebras of C^k-sections by applying Version 2 of the Peetre Theorem.

Theorem 2.74. *Let $k \in \mathbb{N}$. If $\mathfrak{k}$ is perfect or centerfree, then* $\operatorname{Der}\left(\Gamma^k(\mathbb{L})\right) \cong \Gamma^k\left(\operatorname{Der}(\mathbb{L})\right)$ *as Lie algebras.*

Proof. Let D be a derivation of $\Gamma^k(\mathbb{L})$. We want to show that D is automatically a local operator. Let $X \in \Gamma^k(\mathbb{L})$ be zero on an open set $U \subseteq M$.

- Suppose $\mathfrak{z}(\mathfrak{k}) = 0$. Then for all $x \in U$ and $Y \in \Gamma^k(\mathbb{L})$ with $\operatorname{supp}(Y) \subseteq U$ we have $[X, Y] = 0$ and therefore:

$$[(DX)_x, Y_x] = (D\,[X, Y])_x - [X_x, (DY)_x] = 0 - 0 = 0.$$

  By Lemma 2.51, this implies $[(DX)_x, v] = 0$ for all $v \in \mathbb{L}_x$, yielding $(DX)_x \in \mathfrak{z}(\mathbb{L}_x) = 0$ for all $x \in U$.

- Suppose $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$. Since $\Gamma^k(\mathbb{L})$ is also perfect, there exist sections $Y_1, Z_1, \ldots, Y_r, Z_r \in \Gamma^k(\mathbb{L})$ such that $X = \sum_{t=1}^r [Y_t, Z_t]$. Now, if $x \in U$ then there is an x-neigbourhood $W \subseteq U$ and a smooth map $\rho : M \to [0, 1]$ such that $\rho_{|W} \equiv 1$ and $\rho_{M \setminus U} \equiv 0$. We set $X' := (1 - \rho)^2 \cdot X$. Thus $X = X'$ on M and we obtain $X = \sum_{t=1}^r [(1 - \rho) \cdot Y_t, (1 - \rho) \cdot Z_t]$ yielding

$$(DX)_x = \sum_{t=1}^{r} [D((1-\rho)\cdot Y_t)(x), (1-\rho)(x)\cdot Z_t(x)] + [(1-\rho)(x)\cdot Y_t(x), D((1-\rho)\cdot Z_t)(x)] = 0.$$

In both cases we have $(DX)_{|U} \equiv 0$, thus D is local.

By applying Theorem 2.65 and Remark 2.66 we have a linear map $\Psi : \operatorname{Der}\left(\Gamma^k(\mathbb{L})\right) \to \Gamma^k\left(\operatorname{Hom}(\mathbb{L}, \mathbb{L})\right)$ well-defined by $\Psi(D)(X_x) := (DX)_x$ for $X \in \Gamma^k(\mathbb{L})$, $x \in M$. Since we have

$$\Psi(D)\,[X_x, Y_x] = (D\,[X, Y])_x = ([DX, Y])_x + ([X, DY])_x = [\Psi(D)(X_x), Y_x] + [X_x, \Psi(D)(Y_x)]$$

for $X, Y \in \Gamma^k(\mathbb{L})$ and $x \in M$, the image of Ψ is contained in $\Gamma^k(\operatorname{Der}(\mathbb{L}))$. Furthermore, the map Ψ is compatible with the Lie brackets on $\operatorname{Der}\left(\Gamma^k(\mathbb{L})\right)$ and $\Gamma^k(\operatorname{Der}(\mathbb{L}))$:

$$\Psi\,[D, D']\,(X_x) = ([D, D']\,X)_x = (D(D'X))_x - (D'(DX))_x = [\Psi(D), \Psi(D')]\,(X_x).$$

Finally, the function $\Phi : \Gamma^k(\operatorname{Der}(\mathbb{L})) \to \operatorname{Der}\left(\Gamma^k(\mathbb{L})\right)$ defined by $\left(\Phi\left(\mathfrak{D}\right)(X)\right)_x := \mathfrak{D}_x(X_x)$ for a section $\mathfrak{D} \in \Gamma^k(\operatorname{Der}(\mathbb{L}))$, $X \in \Gamma^k(\mathbb{L})$, $x \in M$, is the inverse of Ψ. We conclude that Ψ is an isomorphism of Lie algebras. □

In order to analyze $\Gamma(\mathbb{L}) = \Gamma^\infty(\mathbb{L})$ we define the notion of an x-derivation of $\mathbb{L}$.

Definition 2.75. Let $x \in M$.

1. An *x-derivation* of $\mathbb{L}$ is a linear map $\delta : \Gamma(\mathbb{L}) \to \mathbb{L}_x$ such that the following condition is satisfied for all $X, Y \in \Gamma(\mathbb{L})$:

$$\delta\,[X, Y] = [\delta(X), Y_x] + [X_x, \delta(Y)]\,.$$

   The vector space of all x-derivations of $\mathbb{L}$ is denoted by $\mathcal{D}_x(\mathbb{L})$, the union $\bigcup_{x \in M} \mathcal{D}_x(\mathbb{L})$ is denoted by $\mathcal{D}(\mathbb{L})$ and we have a projection $\overline{p} : \mathcal{D}(\mathbb{L}) \longrightarrow M$ defined by $\overline{p}(\mathcal{D}_x(\mathbb{L})) = \{x\}$ for $x \in M$.

2. Note that $\operatorname{Der}(\mathbb{L})$ can be embedded into $\mathcal{D}(\mathbb{L})$ via the *natural injection* $i : \operatorname{Der}(\mathbb{L}) \longrightarrow \mathcal{D}(\mathbb{L})$, where $D \in \operatorname{Der}(\mathbb{L}_x)$ is mapped to the x-derivation $i(D) : \Gamma(\mathbb{L}) \to \mathbb{L}_x$, $X \mapsto D(X_x)$.

Theorem 2.76. *Let $\mathfrak{k}$ be perfect or centerfree. Then every x-derivation of $\mathbb{L}$ is a differential operator of order at most 1 and the triple $(\mathcal{D}(\mathbb{L}), \overline{p}, M)$ admits a vector bundle structure such that* $\operatorname{Der}(\Gamma(\mathbb{L}))$ *can be naturally identified with* $\Gamma(\mathcal{D}(\mathbb{L}))$*. In addition, there exists a short exact sequence of vector bundles as follows:*

$$0 \longrightarrow \operatorname{Der}(\mathbb{L}) \xrightarrow{i} \mathcal{D}(\mathbb{L}) \xrightarrow{\sigma} TM \otimes \operatorname{Cent}(\mathbb{L}) \longrightarrow 0.$$

Proof. Let $\delta \in \mathcal{D}_x(\mathbb{L})$ be an x-derivation. By Lemma 2.46, every section $X \in \Gamma(\mathbb{L})$ with $j^1_x(X) = 0$ takes the form

$$X = \sum_{i=1}^{r} f_i^2 X_i \tag{7}$$

for functions $f_i \in \mathrm{C}^\infty(M, \mathbb{K})$, $f_i(x) = 0$ and $X_i \in \Gamma(\mathbb{L})$.

- Suppose $\mathfrak{z}(\mathfrak{k}) = 0$. If $Y \in \Gamma(\mathbb{L})$ is an arbitrary smooth section, then:

$$\begin{aligned}
[\delta(X), Y(x)] &= \sum_{i=1}^{r} \left[\delta\left(f_i^2 X_i\right), Y(x)\right] = \sum_{i=1}^{r} \left(\delta\left[f_i^2 X_i, Y\right] - \underbrace{\left[f_i(x)^2 X_i(x), \delta(Y)\right]}_{=0}\right) \\
&= \sum_{i=1}^{r} \delta\left[f_i X_i, f_i Y\right] = \sum_{i=1}^{r} [\delta(f_i X_i), f_i(x) Y(x)] + [f_i(x) X_i(x), \delta(f_i Y)] \\
&= \sum_{i=1}^{r} 0 + 0 = 0,
\end{aligned}$$

  thus $\delta(X) \in \mathfrak{z}(\mathbb{L}_x) = 0$.

- Suppose $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$. Since $\Gamma(\mathbb{L})$ is also perfect, there exist, for each $i \in \{1, \dots, r\}$, sections $Y_{i1}, Z_{i1}, \dots, Y_{is}, Z_{is} \in \Gamma(\mathbb{L})$ such that $X_i = \sum_{t=1}^{s} [Y_{it}, Z_{it}]$. Then:

$$\begin{aligned}
\delta(X) &= \sum_{i=1}^{r} \delta\left(f_i^2 X_i\right) = \sum_{i=1}^{r} \sum_{t=1}^{s} \delta\left[f_i Y_{it}, f_i Z_{it}\right] \\
&= \sum_{i=1}^{r} \sum_{t=1}^{s} [\delta(f_i Y_{it}), f_i(x) Z_{it}(x)] + [f_i(x) Y_{it}(x), \delta(f_i Z_{it})] = \sum_{i=1}^{r} \sum_{t=1}^{s} 0 = 0.
\end{aligned}$$

For both cases, we conclude:

$$\delta(X) = 0 \text{ if } j^1_x(X) = 0. \tag{8}$$

Let (U, φ) be a bundle chart in x and (U, ξ) a corresponding chart of M. Since $\pi : \mathbb{L} \to M$ is defined by an $\mathrm{Aut}(\mathfrak{k})$-valued cocyle, φ restricted to any fiber is an isomorphism of Lie algebras, therefore the Lie bracket on $\Gamma\left(\pi^{-1}(U)\right)$ corresponds to the natural pointwise Lie bracket on $\mathrm{C}^\infty(U, \mathfrak{k})$ meaning $[X, Y]^\varphi = [X^\varphi, Y^\varphi]$ for $X, Y \in \Gamma\left(\pi^{-1}(U)\right)$. Note, by relation (8), that δ has a local form without terms of order greater than 1:

$$X^\varphi \xmapsto{\delta^\varphi} D\left(X^\varphi(x)\right) + \sum_{i=1}^{m} S^i\left(\partial_{\xi_i} X^\varphi(x)\right) \tag{9}$$

for certain $D, S^1, \dots, S^m \in \mathrm{End}(\mathfrak{k})$.

Let us show now that a local form of the type (9) is satisfied for an x-derivation δ^φ if and only if $D \in \mathrm{Der}(\mathfrak{k})$ and $S^1, \dots, S^m \in \mathrm{Cent}(\mathfrak{k})$. Equation (9) implies the following two equations:

$$\begin{aligned}
\delta^\varphi\left[X^\varphi, Y^\varphi\right] &= D\left([X^\varphi(x), Y^\varphi(x)]\right) + \sum_{i=1}^{m} S^i\left(\partial_{\xi_i}\left[X^\varphi, Y^\varphi\right](x)\right) \\
&= D\left([X^\varphi(x), Y^\varphi(x)]\right) + \sum_{i=1}^{m} S^i\left([\partial_{\xi_i} X^\varphi(x), Y^\varphi(x)] + [X^\varphi(x), \partial_{\xi_i} Y^\varphi(x)]\right).
\end{aligned} \tag{10}$$

$$\begin{aligned}
\left[\delta^\varphi\left(X^\varphi\right), Y^\varphi(x)\right] + \left[X^\varphi(x), \delta^\varphi\left(Y^\varphi\right)\right] &= \left[D\left(X^\varphi(x)\right), Y^\varphi(x)\right] + \left[X^\varphi(x), D\left(Y^\varphi(x)\right)\right] \\
&+ \sum_{i=1}^{m} \left[S^i\left(\partial_{\xi_i} X^\varphi(x)\right), Y^\varphi(x)\right] + \left[X^\varphi(x), S^i\left(\partial_{\xi_i} Y^\varphi(x)\right)\right].
\end{aligned} \tag{11}$$

By comparing the equations (10) and (11) for constant sections X, Y we obtain

$$D\left([X^\varphi(x), Y^\varphi(x)]\right) = \left[D\left(X^\varphi(x)\right), Y^\varphi(x)\right] + \left[X^\varphi(x), D\left(Y^\varphi(x)\right)\right],$$

thus D is a derivation of $\mathfrak{k}$. Knowing this we compare the equations (10) and (11) for constant X and arbitrary Y and obtain, after the cancelation of the derivation part:

$$\sum_{i=1}^{m} S^i \left[X^\varphi(x), \partial_{\xi_i} Y^\varphi(x)\right] = \sum_{i=1}^{m} \left[X^\varphi(x), S^i\left(\partial_{\xi_i} Y^\varphi(x)\right)\right].$$

Since for any $i \in \{1, \ldots, m\}$ we have

$$\partial_{\xi_i} Y^\varphi(x) = \left.\frac{d}{dt}\right|_{t=0} Y^\varphi\left(\xi^{-1}(\xi(x) + te_i)\right) = \left(T_x Y^\varphi\right)\left[\left(T_{\xi(x)}\xi^{-1}\right)(e_i)\right]$$

and $T_{\xi(x)}\xi^{-1}$ is a linear bijection, there is, for any $x \in U$ and $v \in \mathfrak{k}$, a smooth map $Y^\varphi : U \to \mathfrak{k}$ with $\partial_{\xi_i} Y^\varphi(x) = \delta_{ij} v$ for $j \in \{1, \ldots, m\}$. Therefore we may conclude that each S^i is contained in the centroid of $\mathfrak{k}$.

We obtain a bijective correspondence:

$$\begin{aligned} \bigcup_{x \in U} \mathcal{D}_x\left(U \times \mathfrak{k}\right) &\longrightarrow U \times \left(\mathrm{Der}(\mathfrak{k}) \times \mathrm{Cent}(\mathfrak{k})^m\right) \\ \delta^\varphi &\longmapsto \left(\overline{p}\left(\delta^\varphi\right), \left(D, \left(S^1, \ldots, S^m\right)\right)\right). \end{aligned} \tag{12}$$

By extending this construction to all charts of a maximal bundle atlas of M, we construct trivializations of $\mathcal{D}(\mathbb{L})$. In order to show that we obtain a bundle structure on $\mathcal{D}(\mathbb{L})$ we will show that the transitions are linear: Let (U, φ) and (V, ψ) be bundle charts of $\pi : \mathbb{L} \to M$ in x with corresponding charts (U, ξ) and (V, η) of M with $\xi(x) = \eta(x) = 0$ and let $D, \widetilde{D} \in \mathrm{Der}(\mathfrak{k})$, $S^1, \ldots, S^m, \widetilde{S}^1, \ldots, \widetilde{S}^m \in \mathrm{Cent}(\mathfrak{k})$ be the endomorphisms of $\mathfrak{k}$ such that for all $X^\varphi, X^\psi \in \mathrm{C}^\infty(U \cap V, \mathfrak{k})$ we have

$$\delta^\varphi\left(X^\varphi\right) = D\left(X^\varphi(x)\right) + \sum_{i=1}^{m} S^i\left(\partial_{\xi_i} X^\varphi(x)\right)$$

and

$$\delta^\psi\left(X^\psi\right) = \widetilde{D}\left(X^\psi(x)\right) + \sum_{i=1}^{m} \widetilde{S}^i\left(\partial_{\eta_i} X^\psi(x)\right).$$

By the definition of the local forms, we have

$$\left(\left(\varphi_{|\mathbb{L}_x}\right)^{-1} \circ \delta^\varphi\right)\left(X^\varphi\right) = \delta(X) = \left(\left(\psi_{|\mathbb{L}_x}\right)^{-1} \circ \delta^\psi\right)\left(X^\psi\right)$$

and

$$\left(\varphi_{|\mathbb{L}_x}\right)^{-1}\left(X^\varphi(x)\right) = X(x) = \left(\psi_{|\mathbb{L}_x}\right)^{-1}\left(X^\psi(x)\right),$$

thus

$$\delta^\psi\left(X^\psi\right) = g(x)\left(\delta^\varphi\left(X^\varphi\right)\right) \text{ and } X^\varphi(x) = g(x)^{-1}\left(X^\psi\right)$$

for a transition map $g : U \cap V \to \mathrm{Aut}(\mathfrak{k})$ of the principal bundle to which $\pi : \mathbb{L} \to M$ is associated. By considering the commutative diagram

$$\begin{tikzcd} \mathfrak{k} = T_{\varphi(x)}\mathfrak{k} \arrow[d, "g(x)"'] & T_x M \arrow[l, "T_x X^\varphi"'] \arrow[r, "T_x\xi"] & \mathbb{R}^m \\ \mathfrak{k} = T_{\psi(x)}\mathfrak{k} & T_x M \arrow[l, "T_x X^\psi"] \arrow[ur, "T_x\eta"'] & \end{tikzcd},$$

we may calcuate

$$\begin{aligned}\partial_{\xi_i} X^\varphi(x) &= T_x X^\varphi\left(T_0\xi^{-1}(e_i)\right) = \left(g(x)^{-1}\circ T_x X^\psi \circ T_0\eta^{-1}\circ T_x\xi\right)\left(T_0\xi^{-1}(e_i)\right)\\ &= \left(g(x)^{-1}\circ T_x X^\psi\right)\left(T_0\eta^{-1}(e_i)\right) = g(x)^{-1}\left(\partial_{\eta_i} X^\psi(x)\right)\end{aligned}$$

and obtain, by the definition of $D, \widetilde{D}, S^1, \ldots, S^m, \widetilde{S}^1, \ldots, \widetilde{S}^m$:

$$\begin{aligned}\widetilde{D}\left(X^\psi(x)\right) + \sum_{i=1}^m \widetilde{S}^i\left(\partial_{\eta_i} X^\psi(x)\right) &= \delta^\varphi\left(X^\varphi\right) = g(x)\left(\delta^\varphi\left(X^\varphi\right)\right) = g(x)\left(D\left(X^\varphi(x)\right) + \sum_{i=1}^m S^i\left(\partial_{\xi_i} X^\varphi(x)\right)\right)\\ &= \left\{g(x)\circ D\circ g(x)^{-1}\right\}\left(X^\psi(x)\right) + \sum_{i=1}^m\left\{g(x)\circ S^i\right\}\left(\partial_{\xi_i} X^\varphi(x)\right)\\ &= \left\{g(x)\circ D\circ g(x)^{-1}\right\}\left(X^\psi(x)\right) + \sum_{i=1}^m\left\{g(x)\circ S^i\circ g(x)^{-1}\right\}\left(\partial_{\eta_i} X^\psi(x)\right).\end{aligned} \tag{13}$$

For contstant sections X this equation implies

$$\widetilde{D}\left(X^\psi(x)\right) = \left\{g(x)\circ D\circ g(x)^{-1}\right\}\left(X^\psi(x)\right),$$

thus $\widetilde{D} = g(x)\circ D\circ g(x)^{-1}$. Knowing this, equation (13) also implies

$$\sum_{i=1}^m \widetilde{S}^i\left(\partial_{\eta_i} X^\psi(x)\right) = \sum_{i=1}^m\left\{g(x)\circ S^i\circ g(x)^{-1}\right\}\left(\partial_{\eta_i} X^\psi(x)\right)$$

and, by inserting appropriate sections for X, we may conclude that $\widetilde{S}^i = g(x)\circ S^i\circ g(x)^{-1}$ for all $i\in\{1,\ldots,m\}$. So the transitions $D\mapsto\widetilde{D}$ and $S^i\to\widetilde{S}^i$ are linear isomorphisms and we have, by Theorem 2.41, a vector bundle $\overline{p}:\mathcal{D}(\mathbb{L})\to M$ with fiber $\mathrm{Der}(\mathfrak{k})\times\mathrm{Cent}(\mathfrak{k})^m$. There is an isomorphism of Lie algebras $\Phi:\mathrm{Der}(\Gamma(\mathbb{L}))\to\Gamma(\mathcal{D}(\mathbb{L}))$, defined as follows: If $D\in\mathrm{Der}(\Gamma(\mathbb{L}))$, then $\Phi(D)$ is the map $M\to\mathcal{D}(\mathbb{L})$, $x\mapsto \mathrm{ev}_x\circ D$. If $\mathfrak{X}\in\Gamma(\mathcal{D}(\mathbb{L}))$, $X\in\Gamma(\mathbb{L})$ and $x\in M$, then $\left(\Phi^{-1}(\mathfrak{X})X\right)_x = (\mathfrak{X}_x)\,X$.

Let us define the map $\sigma:\mathcal{D}(\mathbb{L})\to TM\otimes\mathrm{Cent}(\mathbb{L})$. If $\delta\in\mathcal{D}_x(\mathbb{L})$, then we define $\sigma(\delta)\in T_xM\otimes\mathrm{End}(\mathbb{L}_x)$ as follows: We identify $T_xM\otimes\mathrm{End}(\mathbb{L}_x) = \mathrm{Hom}(T_xM^*,\mathrm{End}(\mathbb{L}_x))$ in the natural way, i.e. the element $\sum_i v_i\otimes A_i$ corresponds to the endomorphism $\alpha\mapsto\sum_i\alpha(v_i)\cdot A_i$. For any $\alpha\in T_xM^*$ and $w\in\mathbb{L}_x$ let $a\in C^\infty(M,\mathbb{R})$ and $A\in\Gamma(\mathbb{L})$ such that $T_xa=\alpha$ and $A(x)=w$. Then $\sigma(\delta)$ is pointwisely defined by

$$\{\sigma(\delta)(\alpha)\}(w) := \delta(aA) - a(x)\delta(A).$$

Now let us check that σ is well-defined and valued in $TM\otimes\mathrm{Cent}(\mathbb{L})$ by using (8) and (12):

$$\begin{aligned}(\{\sigma(\delta)(\alpha)\}(w))^\varphi &= (\delta(aA) - a(x)\delta(A))^\varphi\\ &= D(a(x)A^\varphi(x)) + \sum_{i=1}^m S^i(\partial_{\xi_i}(aA^\varphi)(x)) - a(x)\left(D(A^\varphi(x)) + \sum_{i=1}^m S^i(\partial_{\xi_i}A^\varphi(x))\right)\\ &= \sum_{i=1}^m S^i\left(\partial_{\xi_i}a(x)\cdot A^\varphi(x) + a(x)\partial_{\xi_i}A^\varphi(x) - a(x)\partial_{\xi_i}A^\varphi(x)\right)\\ &= \sum_{i=1}^m S^i\left(\partial_{\xi_i}a(x)\cdot\varphi(w)\right) = \sum_{i=1}^m \partial_{\xi_i}a(x)\cdot S^i\left(\varphi(w)\right).\end{aligned}$$

By leaving the local coordinates we see that $\{\sigma(\delta)(\alpha)\}$ only depends on α and w and we also obtain $\sigma(\delta)\in T_xM\otimes\mathrm{Cent}(\mathbb{L}_x)$. The above calculation also shows that σ maps exactly the elements without local part in $\mathrm{Cent}(\mathfrak{k})^m$ to zero, i.e. $\mathrm{im}\, i = \ker\sigma$.

We finally show the surjectivity of $\sigma : \mathcal{D}(\mathbb{L}) \to TM \otimes \mathrm{Cent}(\mathbb{L})$: Let $x \in M$ and (U, φ) a bundle chart in x with corresponding chart (U, ξ) of M and $\rho : M \to [0,1]$ a smooth map with $\operatorname{supp} \rho \subseteq U$ and $\rho_{|W} \equiv 1$ for an x-neighbourhood $W \subseteq U$ and $(S^1, \ldots, S^m) \in \mathrm{Cent}(\mathfrak{k})^m$. We define $f_i := \left(\varphi_{|\mathbb{L}_x}\right)^{-1} \circ S^i \circ \varphi \in \mathrm{Cent}(\mathbb{L}_x)$ and $\delta \in \mathcal{D}_x(\mathbb{L})$ is defined by $\delta(X) := \rho \cdot \left(\varphi_{|\mathbb{L}_x}\right)^{-1} \left(\sum_{i=1}^m S^i \left(\partial_{\xi_i} X^\varphi(x)\right)\right)$ for $X \in \Gamma(\mathbb{L})$. Then the local part of $\sigma(\delta)$ is $(S^1, \ldots, S^m)$. $\square$

The short exact sequence in Theorem 2.76 naturally induces another short exact sequence.

Corollary 2.77. *The following induced sequence is exact:*

$$0 \longrightarrow \Gamma(\mathrm{Der}\,(\mathbb{L})) \xrightarrow{i} \Gamma(\mathcal{D}(\mathbb{L})) \xrightarrow{\sigma} \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L})) \longrightarrow 0,$$

where $(i(X))(x) := i(X(x))$ *and* $(\sigma(X))(x) := \sigma(X(x))$.

Definition 2.78. The map $\sigma : \mathcal{D}(\mathbb{L}) \to TM \otimes \mathrm{Cent}(\mathbb{L})$ from Theorem 2.76 is called *symbol map*.

For a global description of $\mathrm{Der}(\Gamma(\mathbb{L}))$ we firstly extend the notion of Lie connections. Fix a Lie connection $\nabla : \Gamma(TM) \to \mathrm{Der}(\Gamma(\mathbb{L}))$.

Remark 2.79. The symbol of a derivation ∇_X of $\Gamma(\mathbb{L})$ for a vector field $X \in \Gamma(TM)$ is $X \otimes \mathbf{1}$: Let $x \in M$ be a point, $a \in C^\infty(M, \mathbb{R})$ a smooth map and $A \in \Gamma(\mathbb{L})$ a section of $\mathbb{L}$. Then we identify $\nabla_X \in \mathrm{Der}(\Gamma(\mathbb{L}))$ with $(y \mapsto \mathrm{ev}_y \circ \nabla_X) \in \Gamma(\mathcal{D}(\mathbb{L}))$ and calculate:

$$\begin{aligned}
[\sigma(\nabla_X(x))(T_x a)](A_x) &= (\nabla_X(x))(aA) - a(x)(\nabla_X(x))(A) = (\nabla_X(aA))(x) - a(x)(\nabla_X A)(x) \\
&= ((X.a)A + a\nabla_X(A))(x) - (a\nabla_X A)(x) = ((X.a)A)(x) = (T_x a)(X_x) \cdot A_x
\end{aligned}$$

and therefore

$$\sigma(\nabla_X(x)) = X_x \otimes \mathbf{1}_x.$$

Thus, by the short exact sequence of Corollary 2.77, we see that not all derivations of $\Gamma(\mathbb{L})$ can be described like that. It is necessary to extend ∇ in an appropriate way.

Definition 2.80. The *extension* of ∇ to $\Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$ is the linear map

$$\begin{aligned}
\nabla : \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L})) &\longrightarrow \mathrm{Der}(\Gamma(\mathbb{L})) = \Gamma(\mathcal{D}(\mathbb{L})) \\
\mathcal{Y} &\longmapsto \nabla_{\mathcal{Y}}
\end{aligned}$$

defined as follows: Let $\mathcal{Y} \in \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$ be a section and (U, ξ) a chart of M. Then there are unique elements $S^1, \ldots, S^m \in \Gamma(\mathrm{Cent}(\pi^{-1}(U)))$ such that

$$\mathcal{Y}_{|U} = \sum_{i=1}^m \partial_{\xi_i} \otimes S^i.$$

We define $\nabla_{\mathcal{Y}}$ locally by

$$(\nabla_{\mathcal{Y}})_{|U} := \sum_{i=1}^m S^i \circ \nabla_{\partial_{\xi_i}}. \tag{14}$$

We show that this is well-defined: Let (U, ξ) and (V, η) be two charts of M and $x \in U \cap V$. Then $\mathcal{Y}_x$ takes the form $\sum_{i=1}^m (\partial_{\xi_i})_x \otimes s^i = \sum_{j=1}^m \left(\partial_{\eta_j}\right)_x \otimes r^j$ for endomorphisms $s^1, \ldots, s^m, r^1, \ldots, r^m \in \mathrm{Cent}(\mathbb{L}_x)$. Note that, by the uniqueness of these forms, we can write

$$r^j = \mathcal{Y}_x(T_x \eta_j).$$

By writing the matrix $T_{\eta(x)}\left(\xi \circ \eta^{-1}\right)$ as $A = (a_{ij})$ we find the following relation between the basis vectors $\left(\partial_{\xi_i}\right)_x$ and $\left(\partial_{\eta_j}\right)_x$:

$$\left(\partial_{\eta_j}\right)_x = T_x\left(\xi^{-1} \circ \xi\right) T_{\eta(x)}\eta^{-1}(e_j) = T_{\xi(x)}\xi^{-1}(Ae_j) = T_{\xi(x)}\xi^{-1}\left(\sum_{k=1}^{m} a_{kj}e_k\right) = \sum_{k=1}^{m} a_{kj} T_{\xi(x)}\xi^{-1}(e_k)$$
$$= \sum_{k=1}^{m} a_{kj}\left(\partial_{\xi_k}\right)_x .$$

We write $T_{\xi(x)}\left(\eta \circ \xi^{-1}\right) = A^{-1} = (a^{ij})$ and calculate:

$$\begin{aligned}\sum_{j=1}^{m} r^j \circ \nabla_{\left(\partial_{\eta_j}\right)_x} &= \sum_{j=1}^{m} \mathcal{Y}_x(T_x\eta_j) \circ \sum_{k=1}^{m} a_{kj} \nabla_{\left(\partial_{\xi_k}\right)_x} = \sum_{j,k=1}^{m}\left(\sum_{i=1}^{m} \left(\partial_{\xi_i}\right)_x \otimes s^i\right)(T_x\eta_j) \circ a_{kj}\nabla_{\left(\partial_{\xi_k}\right)_x}\\
&= \sum_{i,j,k=1}^{m} (T_x\eta_j)\left(\left(\partial_{\xi_i}\right)_x\right) \cdot s^i \circ a_{kj}\nabla_{\left(\partial_{\xi_k}\right)_x} = \sum_{i,j,k=1}^{m} a^{ij} \cdot a_{kj} \cdot s^i \circ \nabla_{\left(\partial_{\xi_k}\right)_x}\\
&= \sum_{i=1}^{m} s^i \circ \nabla_{\left(\partial_{\xi_i}\right)_x} .\end{aligned}$$

So we have a well-defined extension.

Lemma 2.81. *The extension of ∇ satisfies the following properties for all $\mathcal{Y} \in \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$, $a \in \mathrm{C}^\infty(M, \mathbb{R})$, $A, B \in \Gamma(\mathbb{L})$, $X \in \Gamma(TM)$, $S \in \Gamma(\mathrm{Cent}(\mathbb{L}))$ and $x \in M$:*

1. $\nabla_{\mathcal{Y}}[A, B] = [\nabla_{\mathcal{Y}}A, B] + [A, \nabla_{\mathcal{Y}}B]$, *i.e.* $\nabla_{\mathcal{Y}} \in \mathrm{Der}(\Gamma(\mathbb{L}))$,
2. $\left(\nabla_{\mathcal{Y}}(aA)\right)_x = \left(\mathcal{Y}_x(T_xa)\right)\{A\} + \left(a\nabla_{\mathcal{Y}}A\right)_x$, *i.e.* $\nabla_{\mathcal{Y}}(aA) = (\mathcal{Y}.a)\{A\} + a\nabla_{\mathcal{Y}}A$, *i.e.* $\sigma\left(\nabla_{\mathcal{Y}}\right) = \mathcal{Y}$,
3. $\nabla_{a\mathcal{Y}}A = a\nabla_{\mathcal{Y}}A$,
4. $\nabla_{X \otimes S}A = S \circ \nabla_X A$. *In particular, we have* $\nabla_{X \otimes \mathbf{1}} = \nabla_X$.

Proof. Let (U, ξ) be a chart of M. We show the properties by concluding locally:

1.
$$\begin{aligned}\nabla_{\mathcal{Y}}[A, B] &= \sum_{i=1}^{m} S^i \circ \nabla_{\partial_{\xi_i}}[A, B] = \sum_{i=1}^{m} S^i \circ \left(\left[\nabla_{\partial_{\xi_i}}A, B\right] + \left[A, \nabla_{\partial_{\xi_i}}B\right]\right)\\
&= \sum_{i=1}^{m} S^i \circ \left[\nabla_{\partial_{\xi_i}}A, B\right] + \sum_{i=1}^{m} S^i \circ \left[A, \nabla_{\partial_{\xi_i}}B\right] = [\nabla_{\mathcal{Y}}A, B] + [A, \nabla_{\mathcal{Y}}B] .\end{aligned}$$

2.
$$\begin{aligned}\left(\nabla_{\mathcal{Y}}(aA)\right)_x &= \sum_{i=1}^{m} S^i(x) \circ \left(\nabla_{\partial_{\xi_i}}aA\right)_x = \sum_{i=1}^{m} S^i(x) \circ \left(\left(\partial_{\xi_i}a\right)A\right)_x + \sum_{i=1}^{m} S^i(x) \circ \left(a\nabla_{\partial_{\xi_i}}A\right)_x\\
&= \mathcal{Y}_x(T_xa)[A] + \left(a\nabla_{\mathcal{Y}}A\right)_x .\end{aligned}$$

3. If $\mathcal{Y} = \sum_{i=1}^{m} \partial_{\xi_i} \otimes S^i$ then $a\mathcal{Y} = \sum_{i=1}^{m} \partial_{\xi_i} \otimes aS^i$, thus:
$$\nabla_{a\mathcal{Y}} = \sum_{i=1}^{m} aS^i \circ \nabla_{\partial_{\xi_i}} = a\sum_{i=1}^{m} S^i \circ \nabla_{\partial_{\xi_i}} = a\nabla_{\mathcal{Y}}.$$

4. If $X = \sum_{i=1}^m X^i \partial_{\xi_i}$ then

$$X \otimes S = \left(\sum_{i=1}^m X^i \partial_{\xi_i} \right) \otimes S = \sum_{i=1}^m \partial_{\xi_i} \otimes X^i S$$

implying

$$\nabla_{X \otimes S} A = \sum_{i=1}^m X^i S \circ \partial_{\xi_i} A = S \circ \nabla_X A.$$

□

Remark 2.82.

1. It is possible to define the extension of a Lie connection $\nabla : \Gamma(TM) \to \mathrm{Der}(\Gamma(\mathbb{L})) = \Gamma(\mathcal{D}(\mathbb{L}))$ without the help of charts of M: For $x \in M$ and $X \in \Gamma(TM)$ the x-derivation $(\nabla_X)_x$ depends on X only via X_x and so we can understand ∇ as a map $TM \to \mathcal{D}(\mathbb{L})$. We now extend ∇ in a natural way to a map $\nabla : TM \otimes \mathrm{Cent}(\mathbb{L}) \to \mathcal{D}(\mathbb{L})$ by mapping $X_x \otimes S_x$ to $S_x \circ (\nabla_X)_x$ (cf. Equation (14)).

2. It is not difficult to show that any linear map $L : \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L})) \to \mathrm{Der}(\Gamma(\mathbb{L}))$ satisfying the four conditions of the preceding lemma is the extension of exactly one Lie connection ∇.

We can now formulate and prove a theorem about the global structure of $\mathrm{Der}(\Gamma(\mathbb{L}))$.

Theorem 2.83. *Fix a Lie connection ∇ on $\mathbb{L}$. If $\mathfrak{k}$ is perfect or centerfree, then a linear function $D : \Gamma(\mathbb{L}) \to \Gamma(\mathbb{L})$ is a derivation of $\Gamma(\mathbb{L})$ if and only if there are sections $\mathcal{Y} \in \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$ and $D_0 \in \Gamma(\mathrm{Der}(\mathbb{L}))$ such that $D = \nabla_{\mathcal{Y}} + i(D_0)$ and this decomposition is unique.*

Proof. Obviously, for any $\mathcal{Y} \in \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$ and $D_0 \in \Gamma(\mathrm{Der}(\mathbb{L}))$ the linear map $\nabla_{\mathcal{Y}} + i(D_0)$ is a derivation of $\Gamma(\mathbb{L})$.

On the other hand, fix $D \in \mathrm{Der}(\Gamma(\mathbb{L})) = \Gamma(\mathcal{D}(\mathbb{L}))$ and set $\mathcal{Y} := \sigma(D) \in \Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$. Since $\sigma(\nabla_{\mathcal{Y}}) = \mathcal{Y}$, the linear map $D - \nabla_{\mathcal{Y}} : \Gamma(\mathbb{L}) \to \Gamma(\mathbb{L})$ has symbol 0 and thus identifies with a differential operator of order zero which is a section of $\mathrm{Hom}(\mathbb{L}, \mathbb{L})$. Since D and $\nabla_{\mathcal{Y}}$ are derivations of $\Gamma(\mathbb{L})$, this section takes its values in $\mathrm{Der}(\Gamma(\mathbb{L}))$. Therefore $D - \nabla_{\mathcal{Y}}$ can be written as $i(D_0)$ for a $D_0 \in \Gamma(\mathrm{Der}(\mathbb{L}))$.

The uniqueness of the decomposition follows from the fact that $\mathcal{Y} = \sigma(D)$ is the only element in $\Gamma(TM \otimes \mathrm{Cent}(\mathbb{L}))$ such that $\sigma(\nabla_{\mathcal{Y}}) = \mathcal{Y}$. □

2.7.3 Centroid

In this subsection we will calculate the centroid of $\Gamma^k(\mathbb{L})$. We begin with a technical lemma.

Lemma 2.84. *Let $X \in \Gamma(\mathbb{L})$ be a section which is zero on an open set $U \subseteq M$ and $x \in U$. Then there exists a compact x-neigbourhood F and there are functions $D_t \in \mathrm{Der}(\Gamma(\mathbb{L}))$, $X_t \in \Gamma(\mathbb{L})$ with $X_{t|F} \equiv 0$ for $t \in \{1, \ldots, r\}$ such that*

$$X = \sum_{t=1}^r D_t(X_t).$$

Proof. Let F be a compact x-neigbourhood contained in U. Since M is paracompact, there exists a Palais cover $\left((V_i, \psi_i, \xi_i, \rho_i)_{i \in J}, (J_t)_{t=1}^r \right)$ refining the open cover $\{U, M \backslash F\}$ of M (cf. Definition 2.44 and

Theorem 2.45). For any $i \in J$ and $Y \in \Gamma\left(\pi^{-1}(V_i)\right)$ recall that $Y^{\psi_i} : \mathbb{R}^m \supseteq \xi_i(V_i) \to \mathfrak{k}$ denotes the mapping such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{L} \supseteq \pi^{-1}(V_i) \arrow[r, "(\xi_i \circ \pi, \psi_i)"] & \xi_i(V_i) \times \mathfrak{k} \\
M \supseteq V_i \arrow[u, "Y"] \arrow[r, "\xi_i"'] & \xi_i(V_i). \arrow[u, "(\mathrm{id}, Y^{\psi_i})"']
\end{tikzcd}$$

The Lie bracket on $\Gamma\left(\pi^{-1}(V_i)\right)$ corresponds to the natural pointwise Lie bracket on $\mathrm{C}^\infty(\xi_i(V_i), \mathfrak{k})$, i.e. $[Y, Z]^{\psi_i} = [Y^{\psi_i}, Z^{\psi_i}]$ for $Y, Z \in \Gamma\left(\pi^{-1}(V_i)\right)$, for all $i \in J$. We set $Y := \rho_i X$ and choose a section $X^i \in \Gamma\left(\pi^{-1}(V_i)\right)$ by claiming $\partial_{(\xi_i)_1}(X^i)^{\psi_i} = Y^{\psi_i}$. With $D^i : \Gamma\left(\pi^{-1}(V_i)\right) \to \Gamma\left(\pi^{-1}(V_i)\right)$ defined by $(D^i)^{\psi_i} := \partial_{(\xi_i)_1}$ one obtains

$$\rho_i X = Y = D^i(X^i).$$

Note that in the case of $\mathrm{supp}(\rho_i) \subseteq U$ we can choose $X^i \equiv 0$ because $X_{|U} \equiv 0$ and each V_i is a subset of exactly one, U or $M \backslash F$. By setting $D_t := \sum_{i \in J_t} D^i$ and $X_t := \sum_{i \in J_t} X^i$ we obtain a well-defined derivation D_t and a well-defined section X_t for any $t \in \{1, \ldots, r\}$ and

$$\sum_{t=1}^r D_t(X_t) = \sum_{t=1}^r \left(\sum_{i \in J_t} D^i \left(\sum_{i \in J_t} X^i \right) \right) = \sum_{t=1}^r \left(\sum_{i \in J_t} \rho_i X \right) = X.$$

□

Theorem 2.85. *Any endomorphism $T : \Gamma(\mathbb{L}) \to \Gamma(\mathbb{L})$ with $T \circ \mathrm{Der}(\Gamma(\mathbb{L})) \subseteq \mathrm{Der}(\Gamma(\mathbb{L}))$ is a differential operator of order zero and can be identified with a smooth section of* $\mathrm{Hom}(\mathbb{L}, \mathbb{L})$.

Proof. We want to show that T is local. Let $X \in \Gamma(\mathbb{L})$ be zero on an open set $U \subseteq M$. Lemma 2.84 yields that there exist $D_1, \ldots, D_r \in \mathrm{Der}(\Gamma(\mathbb{L}))$ and $X_1, \ldots, X_r \in \Gamma(\mathbb{L})$, each X_t zero on an x-neighbouhood F, such that $X = \sum_{t=1}^r D_t(X_t)$. We obtain

$$TX_{|F} = \sum_{t=1}^r ((T \circ D_t)(X_t))_{|F} = \sum_{t=1}^r 0 = 0 \tag{15}$$

because any $D \in \mathrm{Der}(\Gamma(\mathbb{L}))$ can be identified with a section of $\mathcal{D}(\mathbb{L})$ by $D \mapsto X_D$, where $X_D(x) := \mathrm{ev}_x \circ D$, and hence is local. By (15) we conclude that T is local.

Theorem 2.63 yields that T is a differential operator. By Theorem 2.76, any derivation $D \in \mathrm{Der}(\Gamma(\mathbb{L}))$ is a differential operator of order at most one and so is any $T \circ D$. If T were of order $n > 0$, then there would be a bundle chart (U, φ) with corresponding chart (U, ξ) of M and functions $f_\alpha \in \mathrm{C}^\infty(U, \mathrm{End}(\mathfrak{k}))$, $\alpha \in \mathbb{N}^m$, $|\alpha| \leq n$, such that for all $X \in \Gamma\left(\pi^{-1}(U)\right)$ we would have

$$(TX)^\varphi = \sum_{|\alpha| \leq n} f_\alpha \cdot \left(\partial_\xi^\alpha X^\varphi\right)$$

and there would be $x' \in U$, $u \in \mathfrak{k}$ and $\alpha' = (\alpha'_1, \ldots, \alpha'_m) \in \mathbb{N}^m$ with $|\alpha'| = n$ such that $f_{\alpha'}(x')(u) \neq 0$. We may assume, without loss of generality, that $\xi(x') = 0$. By defining $D \in \mathrm{Der}\left(\Gamma\left(\pi^{-1}(U)\right)\right)$ by $D^\varphi := \partial_{\xi_1}$ and $X \in \Gamma\left(\pi^{-1}(U)\right)$ by $X^\varphi(x) := \xi_1(x) \cdot \prod_{i=1}^m \xi_i(x)^{\alpha'_i} \cdot u$ we would obtain

$$(T(DX))^\varphi = \sum_{|\alpha| \leq n} f_\alpha \cdot \left(\partial_\xi^\alpha (DX)^\varphi\right) = \sum_{|\alpha| \leq n} f_\alpha \cdot \underbrace{\left(\partial_\xi^\alpha \partial_{\xi_1} X^\varphi\right)}_{=0 \text{ in } x', \text{ if } \alpha \neq \alpha'}$$
$$\implies (T(DX))^\varphi(x') = f_{\alpha'}(x')\,(\alpha'! \cdot u) \neq 0,$$

contradicting to the facts that $T \circ D$ is a differential operator of order at most one and $j^1_{x'}(X) = 0$. So T is of order zero. □

Theorem 2.86. *If $\mathfrak{k}$ is perfect or centerfree, then* $\operatorname{Cent}\left(\Gamma^k(\mathbb{L})\right) \cong \Gamma^k\left(\operatorname{Cent}(\mathbb{L})\right)$ *as associative algebras.*

Proof. In the smooth case we use Lemma 2.18.2 and Theorem 2.85 to see that all $T \in \operatorname{Cent}(\Gamma(\mathbb{L}))$ identify with smooth sections of $\operatorname{Hom}(\mathbb{L}, \mathbb{L})$.

For $k \in \mathbb{N}$ we will firstly show that any $A \in \operatorname{Cent}(\Gamma^k(\mathbb{L}))$ is local: Let $X \in \Gamma^k(\mathbb{L})$ be zero on an open set $U \subseteq M$.

- Suppose $\mathfrak{z}(\mathfrak{k}) = 0$. Then for all $x \in U$ and $Y \in \Gamma^k(\mathbb{L})$ we have:
$$[(AX)_x, Y_x] = [X_x, (AY)_x] = 0.$$
This implies $(AX)_x \in \mathfrak{z}(\mathbb{L}_x) = 0$ for all $x \in U$.
- Suppose $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$. Then, by Corollary 2.73, the Lie algebra $\Gamma^k(\mathbb{L})$ is also perfect. So there exist sections $Y_1, Z_1, \ldots, Y_r, Z_r \in \Gamma^k(\mathbb{L})$ such that $X = \sum_{t=1}^r [Y_t, Z_t]$. Now, if $x \in U$ then there is an x-neigbourhood $W \subseteq U$ and a smooth map $\rho : M \to [0,1]$ such that $\rho_{|W} \equiv 1$ and $\rho_{M \setminus U} \equiv 0$. We set $X' := (1-\rho) \cdot X$, thus $X = X'$ on M and we obtain $X = \sum_{t=1}^r [Y_t, (1-\rho) \cdot Z_t]$, yielding
$$(AX)_x = \sum_{t=1}^r [(AY_t)(x), (1-\rho)(x) \cdot Z_t(x)] = 0.$$

In both cases we have $(AX)_{|U} \equiv 0$, thus A is local. So we can use Theorem 2.65 to see that all $T \in \operatorname{Cent}(\Gamma^k(\mathbb{L}))$ identify with C^k-sections of $\operatorname{Hom}(\mathbb{L}, \mathbb{L})$.

We know now that, for $k \in \overline{\mathbb{N}}$, there is a linear map $\Psi : \operatorname{Cent}\left(\Gamma^k(\mathbb{L})\right) \to \Gamma^k\left(\operatorname{Hom}(\mathbb{L}, \mathbb{L})\right)$ well-defined by $\Psi(A)(X_x) := (AX)_x$ for $X \in \Gamma^k(\mathbb{L})$, $x \in M$. Since we have
$$\Psi(A)\left[X_x, Y_x\right] = \left(A\left[X, Y\right]\right)_x = \left(\left[AX, Y\right]\right)_x = \left[\Psi(A)(X_x), Y_x\right]$$
for $X, Y \in \Gamma^k(\mathbb{L})$ and $x \in M$, the image of Ψ is contained in $\Gamma^k(\operatorname{Cent}(\mathbb{L}))$. Furthermore, the map Ψ is compatible with the composition of functions on $\operatorname{Cent}\left(\Gamma^k(\mathbb{L})\right)$ and $\Gamma^k(\operatorname{Cent}(\mathbb{L}))$:
$$\Psi(A \circ A')(X_x) = \left((A \circ A')X\right)_x = \left(A(A'X)\right)_x = (\Psi(A) \circ \Psi(A'))(X_x).$$
Finally, the function $\Phi : \Gamma^k(\operatorname{Cent}(\mathbb{L})) \to \operatorname{Cent}\left(\Gamma^k(\mathbb{L})\right)$ defined by $\left(\Phi\left(\mathfrak{A}\right)(X)\right)_x := \mathfrak{A}_x(X_x)$ for a section $\mathfrak{A} \in \Gamma^k(\operatorname{Cent}(\mathbb{L}))$, $X \in \Gamma^k(\mathbb{L})$, $x \in M$, is the inverse of Ψ. We conclude that Ψ is an isomorphism of associative algebras. $\square$

We can now deduce an important structure theorem.

Theorem 2.87. *Let $\mathfrak{k}$ be indecomposable. Then $\Gamma^k(\mathbb{L})$ is so, if* $\operatorname{Hom}(\mathfrak{k}/\left[\mathfrak{k}, \mathfrak{k}\right], \mathfrak{z}(\mathfrak{k})) = 0$.

Proof. Let $\Gamma^k(\mathbb{L}) = I_1 \oplus I_2$ be a decomposition into a direct sum of ideals and P^1 be the projection onto I_1 parallel to I_2. Note that $P^1 \in \operatorname{Cent}(\Gamma^k(\mathbb{L})) = \Gamma^k(\operatorname{Cent}(\mathbb{L}))$ because for any $X, Y \in \Gamma^k(\mathbb{L})$ decomposing into $X_1, Y_1 \in I_1$ and $X_2, Y_2 \in I_2$ we have:
$$P^1\left[X, Y\right] = P^1([X_1, Y_1] + [X_2, Y_2]) = [X_1, Y_1] = [X_1 + X_2, Y_1] = \left[X, P^1 Y\right]. \tag{16}$$
Since $P^1 \circ P^1 = P^1$ on $\Gamma^k(\mathbb{L})$, each $P^1_x \in \operatorname{Cent}(\mathbb{L}_x)$, where $x \in M$, is a projection in $\operatorname{Cent}(\mathbb{L}_x)$ and, by indecomposability of $\mathbb{L}_x \cong \mathfrak{k}$, this implies $P^1_x = 0$ or $P^1_x = \mathbf{1}_x$. The map $M \to \operatorname{Cent}(\mathbb{L}_x) \subseteq \operatorname{Cent}(\mathbb{L})$, $x \mapsto P^1_x$ is continuous and for each bundle atlas $(U_i, \varphi_i)_{i \in I}$, where each U_i is connected, the mapping $x \longmapsto \varphi_i \circ P^1_x$ has discrete image. But all this is only possible if $P^1 = 0$ or $P^1 = \mathbf{1}$, i.e. $I_1 = 0$ or $I_2 = 0$. So $\Gamma^k(\mathbb{L})$ is indecomposable. $\square$

Remark 2.88. If $\operatorname{Hom}(\mathfrak{k}/[\mathfrak{k},\mathfrak{k}],\mathfrak{z}(\mathfrak{k})) = 0$, then, by Lemma 2.18.3, we have $\operatorname{Cent}(\mathfrak{k}) = \mathrm{N}(\mathfrak{k}) \oplus \mathrm{S}(\mathfrak{k})$ and, since $\operatorname{Aut}(\mathfrak{k})$ acts on $\mathrm{N}(\mathfrak{k})$ and $\mathrm{S}(\mathfrak{k})$ by conjugation, we also have the following decomposition: $\Gamma^k(\operatorname{Cent}(\mathbb{L})) = \Gamma^k(\mathrm{N}(\mathbb{L})) \oplus \Gamma^k(\mathrm{S}(\mathbb{L}))$, where $\mathrm{N}(\mathbb{L})$, $\mathrm{S}(\mathbb{L})$ denote the corresponding subbundles $\mathbb{L}(\mathrm{N}(\mathfrak{k}))$, $\mathbb{L}(\mathrm{S}(\mathfrak{k}))$ of $\operatorname{Cent}(\mathbb{L}) = \mathbb{L}(\operatorname{Cent}(\mathfrak{k}))$ (cf. Definition 2.53.2), respectively. Consider the case $\mathrm{S}(\mathfrak{k}) = \mathbb{K}\cdot\mathbf{1}$. Then $\mathrm{S}(\mathbb{L})$ is a bundle over M with fiber $\mathbb{K}\cdot\mathbf{1}$ and, for any $x \in M$, the fiber $\mathrm{S}(\mathbb{L})_x$ is naturally isomorphic to $\mathrm{S}(\mathbb{L}_x)$. Thus any section $X \in \Gamma^k(\mathrm{S}(\mathbb{L}))$ takes the form $M \to \mathrm{S}(\mathbb{L})$, $x \mapsto X_x = f(x)\mathbf{1}_x$ for some $f \in \mathrm{C}^k(M,\mathbb{K})$, hence:

$$\Gamma^k(\mathrm{S}(\mathbb{L})) = \mathrm{C}^k(M,\mathbb{K})\cdot\mathbf{1}. \tag{17}$$

Remark 2.89. Let $\mathfrak{k}$ be reductive and $\dim\mathfrak{z}(\mathfrak{g}) \leq 1$, e.g., $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{K})$ or $\operatorname{Hom}(\mathfrak{k}/[\mathfrak{k},\mathfrak{k}],\mathfrak{z}(\mathfrak{k})) = 0$. If $\mathfrak{k} = \bigoplus_{i=1}^n \mathfrak{k}_i$ is a decomposition into a direct sum of indecomposable non-zero ideals, then we define $\mathrm{H}(\mathfrak{k}) := \bigcap_{i=1}^n \{ f \in \operatorname{Aut}(\mathfrak{k}) | \, f(\mathfrak{k}_i) = \mathfrak{k}_i\}$, the normal subgroup of $\operatorname{Aut}(\mathfrak{k})$ stabilizing this decomposition. This is well-defined because the decomposition into a direct sum of indecomposable non-zero ideals is, by Proposition 2.25, unique except for the order. We denote the quotient group $\operatorname{Aut}(\mathfrak{k})/\mathrm{H}(\mathfrak{k})$ by $\mathrm{G}(\mathfrak{k})$.

Let $\|\cdot\| : \operatorname{End}(\mathfrak{k}) \to \mathbb{R}_+$ be the Frobenius norm on $\operatorname{End}(\mathfrak{k})$ with respect to a basis $(v_1,\dots,v_d)$ corresponding to the decomposition $\mathfrak{k} = \bigoplus_{i=1}^n \mathfrak{k}_i$, i.e. $\|f\| := \sqrt{\sum_{i,j=1}^d a_{ij}^2}$, where $f(v_j) = a_{ij}v_i$ for $i,j \in \{1,\dots,d\}$. This norm induces, by $\dim\mathfrak{k} = d < \infty$, the compact-open topology. So the subspace $\operatorname{Aut}(\mathfrak{k}) \subseteq \operatorname{End}(\mathfrak{k})$ becomes a topological group.

By Remark 2.27, the matrices corresponding to the automorphisms in $\operatorname{Aut}(\mathfrak{k})$ are "permutated block diagonal matrices". Therefore, if $f \in \mathrm{H}(\mathfrak{k})$ with $f(v_j) = a_{ij}v_i$ for $i,j \in \{1,\dots,d\}$ and we set $\varepsilon := 0.5\cdot\min_{i,j\in\{1,\dots,d\}} a_{ij}$, then the ε-ball around f in $\operatorname{Aut}(\mathfrak{k})$ with respect to $\|\cdot\|$ is entirely contained in $\mathrm{H}(\mathfrak{k})$. So $\mathrm{H}(\mathfrak{k}) \subseteq \operatorname{Aut}(\mathfrak{k})$ is open and $\mathrm{G}(\mathfrak{k})$ equipped with the corresponding quotient topology is discrete.

The following general theorem about first Čech cohomology sets is Theorem V.5 of [Ne04]. We will use it to show Lemma 2.91 which will be useful for two structure theorems.

Theorem 2.90. *If M is a smooth manifold and $q : A \to G$ is a smooth and surjective morphism of Lie groups with kernel H, then there exists an exact sequence of morphisms of pointed sets as follows:*

$$1 \longrightarrow \mathrm{C}^\infty(M,H) \longrightarrow \mathrm{C}^\infty(M,A) \longrightarrow \mathrm{C}^\infty(M,G) \longrightarrow \check{\mathrm{H}}^1(M,H) \longrightarrow \check{\mathrm{H}}^1(M,A) \longrightarrow \check{\mathrm{H}}^1(M,G).$$

Lemma 2.91. *Let A be a Lie group, $H \trianglelefteq A$ open and $G := A/H$ such that $\check{\mathrm{H}}^1(M,G) = 1$. Then $\check{\mathrm{H}}^1(M,A) \cong \check{\mathrm{H}}^1(M,H)$.*

Proof. Since $\check{\mathrm{H}}^1(M,G) = 1$, the map $\check{\mathrm{H}}^1(M,H) \to \check{\mathrm{H}}^1(M,A)$ is, by Theorem 2.90, surjective. □

Theorem 2.92. *Let $\mathfrak{k} = \bigoplus_{i=1}^n \mathfrak{k}_i$ be a decomposition into a direct sum of indecomposable non-zero ideals of a perfect or centerfree Lie algebra $\mathfrak{k}$ and let $\check{\mathrm{H}}^1(M,\mathrm{G}(\mathfrak{k}))$ be trivial (e.g. M simply connected). Then there is a decomposition into a direct sum of indecomposable non-zero ideals $\Gamma^k(\mathbb{L}) = \bigoplus_{i=1}^n \Gamma^k(\mathbb{L}^i)$, where each $\pi_{|\mathbb{L}^i} : \mathbb{L}^i \to M$ is a subbundle of $\pi : \mathbb{L} \to M$ with fiber $\mathfrak{k}_i$ and this decomposition is unique except for the order.*

Proof. Lemma 2.91 yields $\check{\mathrm{H}}^1(M,\operatorname{Aut}(\mathfrak{k})) \cong \check{\mathrm{H}}^1(M,\mathrm{H}(\mathfrak{k}))$, so the bundle $\pi : \mathbb{L} \to M$ admits a cocycle in $\mathrm{H}(\mathfrak{k})$ and we can construct the subbundles $\pi_{|\mathbb{L}^i} : \mathbb{L}^i \to M$ with fiber $\mathfrak{k}_i$. We also have

$$\Gamma^k(\mathbb{L}) = \bigoplus_{i=1}^n \Gamma^k(\mathbb{L}^i),$$

which is a decomposition into a direct sum of indecomposable non-zero ideals because each $\mathfrak{k}_i$ is indecomposable (cf. Theorem 2.87). We conclude by showing the uniqueness of this decomposition (except for the order).

Let $\Gamma^k(\mathbb{L}) = I_1 \oplus I_2$ be a decomposition into a direct sum of ideals and P^1 be the projection onto I_1 parallel to I_2 and for $i \in \{1, \ldots, n\}$ let P_i be the projection onto $\Gamma^k(\mathbb{L}^i)$ associated to the decomposition $\Gamma^k(\mathbb{L}) = \bigoplus_{i=1}^n \Gamma^k(\mathbb{L}^i)$. Calculations totally analogous to (16) show that P^1 and each P_i are in $\mathrm{Cent}(\Gamma^k(\mathbb{L})) = \Gamma^k(\mathrm{Cent}(\mathbb{L}))$. For all $x \in M$ we have $\left[P_i, P^1\right](x) = \left[P_i(x), P^1(x)\right] = 0$ because $\mathrm{Cent}(\mathbb{L}_x)$ is abelian by $\mathrm{Hom}(\mathfrak{k}/\left[\mathfrak{k},\mathfrak{k}\right], \mathfrak{z}(\mathfrak{k})) = 0$ and Lemma 2.18.3. Therefore P^1 commutes with P_i for any $i \in \{1, \ldots, n\}$. This implies $P^1\left(\Gamma^k(\mathbb{L}^i)\right) = P_i(I_1)$ and, due to the indecomposability of the ideals, $P^1_{|\Gamma^k(\mathbb{L}^i)} = 0$ or $P^1_{|\Gamma^k(\mathbb{L}^i)} = \mathbf{1}$ for any $i \in \{1, \ldots, n\}$, resulting in

$$I_1 = \bigoplus_{\ell=1}^{t} \Gamma^k\left(\mathbb{L}^{j_\ell}\right)$$

for some $1 \leq j_1 < \ldots < j_t \leq n$. Thus the decomposition $\Gamma^k(\mathbb{L}) = \bigoplus_{i=1}^n \Gamma^k(\mathbb{L}^i)$ is unique. □

A statement concerning complex structures of real Lie algebras of C^k-sections is given by the following proposition.

Proposition 2.93. *Let $\mathbb{K} = \mathbb{R}$ and $\mathfrak{k}$ be indecomposable with $\mathrm{Hom}(\mathfrak{k}/\left[\mathfrak{k},\mathfrak{k}\right], \mathfrak{z}(\mathfrak{k})) = 0$. Then $\Gamma^k(\mathbb{L})$ admits at most two complex structures and, if $\check{\mathrm{H}}^1(M, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z})$ is trivial (e.g. M simply connected), then $\Gamma^k(\mathbb{L})$ admits a complex structure if and only if $\mathfrak{k}$ does.*

Proof. A complex structure $J \in \mathrm{Cent}\left(\Gamma^k(\mathbb{L})\right) = \Gamma^k(\mathrm{Cent}(\mathbb{L}))$ induces a complex structure J_x on each fiber $\mathrm{Cent}(\mathbb{L}_x)$, thus on $\mathfrak{k}$. Lemma 2.28.2 implies that $\mathfrak{k}$ possesses at most two complex structures, thus, by the connectedness of M, there can be at most two different C^k-complex structures, namely J and $-J$, on $\Gamma^k(\mathbb{L})$.

Let J_0 be a complex structure on $\mathfrak{k}$. We turn $\mathfrak{k}$ into a complex Lie algebra by defining the multiplication by a complex scalar via $(a+bi)\cdot x := ax + bJ_0(x)$ for $a, b \in \mathbb{R}$ and $x \in \mathfrak{k}$. If $\sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{k})$, $a, b \in \mathbb{R}$ and $x \in \mathfrak{k}$, then

$$\sigma((a+bi)\cdot x) = \sigma(ax + bJ_0(x)) = a\sigma(x) + b\sigma(J_0(x)).$$

Thus $\sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{k})$ is in $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{k})$ if and only if $\sigma \circ J_0 = J_0 \circ \sigma$. Otherwise we have $\sigma \circ J_0 \circ \sigma^{-1} = -J_0$ because $\left(\sigma \circ J_0 \circ \sigma^{-1}\right)^2 = -\mathbf{1}$ and J_0, $-J_0$ are the only complex structures on $\mathfrak{k}$ by Lemma 2.28.2. So $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{k})$ is a subgroup of index 2 of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{k})$, thus normal and $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{k}) / \mathrm{Aut}_{\mathbb{C}}(\mathfrak{k}) \cong \mathbb{Z}/2\mathbb{Z}$. Since $\check{\mathrm{H}}^1(M, \mathbb{Z}/2\mathbb{Z}) = 1$, we have $\check{\mathrm{H}}^1(M, \mathrm{Aut}_{\mathbb{R}}(\mathfrak{k})) \cong \check{\mathrm{H}}^1(M, \mathrm{Aut}_{\mathbb{C}}(\mathfrak{k}))$ by Lemma 2.91, yielding the existence of an $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{k})$-valued cocyle of $\mathbb{L}$ and this turns $\pi : \mathbb{L} \to M$ into a bundle of Lie algebras with complex fiber $\mathfrak{k}$ and $\Gamma^k(\mathbb{L})$ into a complex Lie algebra.

Conversely, if $\Gamma^k(\mathbb{L})$ admits a complex structure, then, by Theorem 2.86, this induces a complex structure on each fiber $\mathfrak{k}$, thus on $\mathfrak{k}$. □

2.7.4 Isomorphisms

Now we will discuss the isomorphisms of Lie algebras of C^k-sections. In the light of Theorem 2.92 it suffices to do this for indecomposable Lie algebras only. The main key to the analysis of the isomorphisms is Lemma 2.96. We want to show some lemmas before. In this subsection M, N are smooth manifolds with positive dimensions m, n, respectively

Lemma 2.94. *For $k \in \overline{\mathbb{N}}$ the topology on M is the same as the initial topology of the maps in $\mathrm{C}^k(M, \mathbb{K})$.*

Proof. Let $\mathcal{O}_M$ denote the topology on M. The initial topology of the maps in $\mathrm{C}^k(M, \mathbb{K})$, denoted by $\mathcal{O}_i$, is the coarsest topology on M such that each $f \in \mathrm{C}^k(M, \mathbb{K})$ is continuous, so that $\mathcal{O}_i \subseteq \mathcal{O}_M$. Let V be a neighbourhood of $x \in M$ with respect to $\mathcal{O}_M$. Then, by paracompactness of M, there is a smooth function $\rho : M \to [0,1] \subseteq \mathbb{K}$ such that $\rho_{|M\setminus V} \equiv 0$ and $\rho_{|W} \equiv 1$ for a smaller x-neighbourhood $W \subseteq V$.

But this yields $W \subseteq \rho^{-1}(B(0.5;1)) \subseteq V$, where $B(r;z)$ denotes the open ball with radius r around z in $\mathbb{K}$. Thus V is also a neighbourhood of $x \in M$ with respect to $\mathcal{O}_i$ and we conclude $\mathcal{O}_M \subseteq \mathcal{O}_i$, so $\mathcal{O}_i = \mathcal{O}_M$. $\square$

Lemma 2.95. *There exists a non-constant proper map $\varphi \in \mathrm{C}^\infty(M,\mathbb{K})$, i.e. φ-preimages of compact subspaces are compact.*

Proof. Let $U \subsetneq M$ be a non-empty open subset with compact closure, $K \subseteq U$ a non-empty compact subset and $\varphi \in \mathrm{C}^\infty(M,[0,1]) \subseteq \mathrm{C}^\infty(M,\mathbb{K})$ a map such that $\operatorname{supp}\rho \subseteq U$ and $\rho_{|K} \equiv 1$. For any compact $C \subseteq \mathbb{K}$, the set $\varphi^{-1}(C)$ is a closed subset of $\overline{U}$, hence compact. $\square$

Lemma 2.96. *Let $v : \mathrm{C}^k(M,\mathbb{K}) \to \mathrm{C}^\ell(N,\mathbb{K})$ be an isomorphism of associative algebras for some $0 \neq k,\ell \in \overline{\mathbb{N}}$. Then $k = \ell$ and there is a C^k-diffeomorphism $\lambda : M \to N$ such that $v(f) = f \circ \lambda^{-1}$ for all $f \in \mathrm{C}^k(M,\mathbb{K})$.*

Proof. We set $\mathcal{A} := \mathcal{A}^M := \mathrm{C}^k(M,\mathbb{K})$ and $\mathcal{A}_U := \mathcal{A}_U^M := \{\, f \in \mathcal{A} |\, \operatorname{supp} f \subseteq U\}$ for open $U \subseteq M$. Furthermore, for $x_0 \in M$ we write $\mathcal{N}_{x_0} := \mathcal{N}_{x_0}^M := \{\, f \in \mathcal{A} |\, f(x_0) = 0\}$. The symbols $\mathcal{A}^N$, $\mathcal{A}_V^N$ for open $V \subseteq N$ and $\mathcal{N}_{y_0}^N$ for $y_0 \in N$ are understood in the obvious analogous manner.

Since any $\mathcal{N}_{x_0}$ obviously is a vector subspace of $\mathcal{A}$ and $\mathcal{A} \cdot \mathcal{N}_{x_0} \subseteq \mathcal{N}_{x_0}$, it is even an ideal of $\mathcal{A}$. Its codimension[12] is 1 because, evidently, we have $\mathcal{N}_{x_0} \oplus \mathbb{K} \cdot \mathbf{1} = \mathcal{A}$. We will show that in fact *every* ideal $I \trianglelefteq \mathcal{A}$ of codimension 1 takes the form $\mathcal{N}_x$ for a some $x \in M$.[13] For this, it suffices to show $I \subseteq \mathcal{N}_x$ for some $x \in M$ because all $\mathcal{N}_x$ are ideals of codimension 1. Since, for $x_0 \neq x_1$, we have $(\mathcal{N}_{x_0} \cap \mathcal{N}_{x_1}) \oplus \mathbb{K} \cdot \mathbf{1} \neq \mathcal{A}$, the point $x \in M$ corresponding to the ideal I is uniquely determined.

Assume there exists an ideal $I \trianglelefteq \mathcal{A}$ of codimension 1 such that for all $x \in M$ we have $I \not\subseteq \mathcal{N}_x$, i.e., there exists $f_0 \in I$ with $f_0(x) \neq 0$. By continuity, there is even an admissible[14] open x-neighbourhood $U \subseteq M$ with $f_0(x') \neq 0$ for all $x' \in U$. Note that M can, in our case, be covered by admissible open sets, say $M = \bigcup_{\gamma \in \Gamma} U_\gamma$. If $U \subseteq M$ is open and $f \in \mathcal{A}_U$ then $\frac{f}{f_0}$ is a well-defined function in $\mathcal{A} = \mathrm{C}^k(M,\mathbb{K})$ and $f = f_0 \cdot \frac{f}{f_0} \in I$ because $f_0 \in I$ and I is an ideal of $\mathcal{A}$. Thus $\mathcal{A}_U \subseteq I$.

Now let $\varphi \in \mathcal{A}$ be a non-locally constant proper map, which exists by Lemma 2.95. Since $\operatorname{codim}_{\mathcal{A}} I = 1$ and $\dim\left(\mathbb{K} \cdot \varphi \oplus \mathbb{K} \cdot \varphi^2\right) = 2$, there are $a, b \in \mathbb{K}$ such that $a\varphi + b\varphi^2 =: f_1 \in I$ is not zero and $f_1^{-1}(\{0\})$ is compact because $f_1^{-1}(\{0\}) = (b\varphi)^{-1}(\{0\}) \cup \varphi^{-1}\left(\left\{-\frac{a}{b}\right\}\right)$ and $b\varphi$ is proper in the case of $b \neq 0$ and $f_1^{-1}(\{0\}) = (a\varphi)^{-1}(\{0\})$ and $a\varphi$ is proper if $b = 0$. We define an open set $U_1 := f_1^{-1}(\mathbb{K} \setminus \{0\})$. The compact subset $f_1^{-1}(\{0\}) \subseteq M$ is covered by finitely many admissible open sets $U_2, U_3, \ldots, U_r \in (U_\gamma)_{\gamma \in \Gamma}$. Let $(V_\beta)_{\beta \in B}$ be a locally finite refinement of $(U_1, \ldots, U_r)$ such that there is a smooth partition of unity $(\rho_\beta : M \to [0,1])_{\beta \in B}$ subordinate to $(V_\beta)_{\beta \in B}$. Let $B_1, \ldots, B_r \subseteq B$ be subsets such that $B = \bigcup_{i=1}^r B_i$ and let $\beta \in B_i$ for some $i \in \{1, \ldots, r\}$ imply $V_\beta \subseteq U_i$. Then, for any $i \in \{1, \ldots r\}$ and $\beta \in B_i$, we have $\rho \cdot \mathcal{A} \subseteq \mathcal{A}_{U_i}$. We obtain

$$\mathcal{A} = \mathbf{1} \cdot \mathcal{A} = \sum_{\beta \in B} (\rho_\beta \cdot \mathcal{A}) \subseteq \sum_{i=1}^r \left(\sum_{\beta \in B_i} \rho_\beta \cdot \mathcal{A} \right) \subseteq \sum_{i=1}^r \mathcal{A}_{U_i} \subseteq I,$$

but this contradicts the fact that I has codimension 1. This shows that for any ideal $I \trianglelefteq \mathcal{A}$ with codimension 1 there exists $x \in M$ such that $I = \mathcal{N}_x$.

Since v is an isomorphism of associative algebras, $v(\mathcal{N}_x)$ is, for any $x \in M$, an ideal of $\mathcal{A}^N$ with codimension 1, thus equal to $\mathcal{N}_y^N$ for some unique $y \in N$. We write $y =: \lambda(x)$ and obtain a mapping $\lambda : M \to N$. A dual argumentation with v^{-1} instead of v leads to a mapping $\widetilde{\lambda} : N \to M$ such that

[12] The "codimension" of an ideal $I \trianglelefteq \mathcal{A} = \mathcal{A}^M$ or $J \trianglelefteq \mathcal{A}^N$ is always meant to be the vector space dimension of the quotient algebra $\mathcal{A}/I$ or $\mathcal{A}^N/J$, respectively.

[13] The proof will also show the analogous statemanet for the ideals $J \trianglelefteq \mathcal{A}^N$ with codimension 1.

[14] We call an open set U *admissible*, if there is a function $f \in I$ such that $f(y) \neq 0$ for all $y \in U$.

$\lambda \circ \widetilde{\lambda} = \mathrm{id}_N$ and $\widetilde{\lambda} \circ \lambda = \mathrm{id}_M$, so λ is bijective. Note that we can perform the following calculation for $f \in \mathcal{A}$ and $y \in N$:

$$f\left(\lambda^{-1}(y)\right) = 0 \quad \Longrightarrow \quad f \in \mathcal{N}_{\lambda^{-1}(y)} \quad \Longrightarrow \quad v(f) \in \mathcal{N}_y^N \quad \Longrightarrow \quad v(f)(y) = 0.$$

But this already implies that $f \circ \lambda^{-1} = v(f)$ for any $f \in \mathcal{A}$ because we may replace the mapping f by $f - r \cdot \mathbf{1}$ for arbitrary $r \in \mathbb{K}$ in the above calculation. We will now show that λ is a C^k-diffeomorphism.

By Lemma 2.94, the topologies on M and N can be described as the initial topologies of the maps in $\mathrm{C}^k(M,\mathbb{K})$ and $\mathrm{C}^\ell(N,\mathbb{K})$, respectively. Since $v^{-1}(g) = g \circ \lambda \in \mathrm{C}^k(M,\mathbb{K})$ for all $g \in \mathrm{C}^\ell(N,\mathbb{K})$, the map $\lambda : M \to N$ is continuous. Let $(U_i', \varphi_i')_{i\in I}$ be a locally finite atlas of M and $(V_j', \psi_j')_{j\in J}$ a locally finite atlas of N. We modify the chart maps by multiplying them with the maps $\rho_i : M \to [0,1]$ and $\varpi_j : N \to [0,1]$, respectively, of smooth partitions of unity subordinate to the atlases. Then $\rho_{i|U_i} \equiv 1$ and $\varpi_{j|V_j} \equiv 1$ for open sets $U_i \subseteq U_i'$ and $V_j \subseteq V_j'$ still covering the whole manifold. We obtain new atlases, denoted by $(U_i, \varphi_i)_{i\in I}$ and $(V_j, \psi_j)_{j\in J}$, where each chart map is defined on the whole manifold. Now let $t \in \{1, \dots, n\}$, $i \in I$, $j \in J$ and define $A := \varphi_i\left(U_i \cap \lambda^{-1}(V_i)\right) \subseteq \mathbb{R}^m$ and $D := e_t^*(\psi_j(V_j \cap \lambda(U_i))) \subseteq \mathbb{R}$. The map $e_t^* \circ \psi_j \circ \lambda \circ (\varphi_i)^{-1} : A \to D$ is equal to

$$\underbrace{v^{-1}\left(e_t^* \circ \psi_j\right)}_{\in \mathrm{C}^k(M,\mathbb{R})} \circ (\varphi_i)^{-1} \in \mathrm{C}^k(A, D),$$

thus, since t,i and j were arbirarily chosen, $\lambda : M \to N$ is C^k.

By the symmetry of the arguments, λ^{-1} is a C^ℓ-map and, by the Inverse Mapping Theorem, even a $\max(k,\ell)$-times continuously differentiable diffeomorphism. Since the composition with λ turns $\mathrm{C}^\ell(N,\mathbb{K})$-maps into $\mathrm{C}^k(M,\mathbb{K})$-maps and λ^{-1} turns $\mathrm{C}^k(M,\mathbb{K})$-maps into $\mathrm{C}^\ell(N,\mathbb{K})$-maps, so $\mathrm{C}^k(M,\mathbb{K}) = \mathrm{C}^\ell(M,\mathbb{K})$ and $\mathrm{C}^k(N,\mathbb{K}) = \mathrm{C}^\ell(N,\mathbb{K})$, thus $k = \ell$, completing the proof. □

The following Lemma will be used in Theorem 2.98.

Lemma 2.97.

1. *Let V be a vector space of finite dimension, (U,ξ) a chart of M and $T : U \to \mathrm{End}(V)$, $f : U \to V$ smooth functions. Then, for any multi-index $\alpha \in \mathbb{N}^m$, we have:*

$$\partial_\xi^\alpha (T \cdot f) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \left(\partial_\xi^\gamma T \cdot \partial_\xi^{\alpha-\gamma} f\right). \tag{18}$$

2. *We have, for $\alpha \in \mathbb{N}^m$:*

$$\sum_{\gamma\le\alpha} \frac{(-1)^{|\alpha-\gamma|}}{\gamma!(\alpha-\gamma)!} = \begin{cases} 1, \text{ if } \alpha = 0 \\ 0 \text{ otherwise} \end{cases} = \frac{\delta_{\alpha,0}}{\alpha!}. \tag{19}$$

Proof.

1. The proof is done by mathematical induction over $|\alpha|$. The claim is trivially true for $\alpha = 0$. For $|\alpha| = 1$ there exists a canonical basis vector e_i of $\mathbb{R}^m$ such that $\alpha = e_i$ and $\partial_\xi^\alpha = \partial_{\xi_i}$. Then

$$\partial_{\xi_i}(T \cdot f) = \partial_{\xi_i} T \cdot f + T \cdot \partial_{\xi_i} f$$

and the claim is also true. Now consider the claim shown for all multi-indices $\beta \in \mathbb{N}^m$ with $|\beta| = n > 0$ and fix $\alpha \in \mathbb{N}^m$ with $|\alpha| = n+1$. There exists a canonical basis vector e_i of $\mathbb{R}^m$ such that $\beta := \alpha - e_i \in \mathbb{N}^m$ and $|\beta| = n$. By induction hypothesis, we have

$$\partial_\xi^\beta (T \cdot f) = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial_\xi^\gamma T \cdot \partial_\xi^{\beta-\gamma} f,$$

thus

$$\begin{aligned}
\partial_\xi^\alpha(T\cdot f) &= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial_{\xi_i}\left(\partial_\xi^\gamma T\cdot\partial_\xi^{\beta-\gamma}f\right) = \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\left(\partial_\xi^{\gamma+e_i}T\cdot\partial_\xi^{\beta-\gamma}f + \partial_\xi^\gamma T\cdot\partial_\xi^{\beta-\gamma+e_i}f\right)\\
&= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial_\xi^{\gamma+e_i}T\cdot\partial_\xi^{\beta-\gamma}f + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial_\xi^\gamma T\cdot\partial_\xi^{\beta-\gamma+e_i}f\\
&= \sum_{e_i\le\delta\le\alpha}\binom{\alpha-e_i}{\delta-e_i}\partial_\xi^\delta T\cdot\partial_\xi^{\alpha-\delta}f + \sum_{\gamma\le\alpha-e_i}\binom{\alpha-e_i}{\gamma}\partial_\xi^\gamma T\cdot\partial_\xi^{\alpha-\gamma}f\\
&= 1\cdot\partial_\xi^0 T\cdot\partial_\xi^\alpha f + \sum_{e_i\le\delta\le\alpha-e_i}\left(\binom{\alpha-e_i}{\delta-e_i}+\binom{\alpha-e_i}{\delta}\right)\partial_\xi^\delta T\cdot\partial_\xi^{\alpha-\delta}f + 1\cdot\partial_\xi^\alpha T\cdot\partial_\xi^0 f\\
&= \sum_{\delta\le\alpha}\binom{\alpha}{\delta}\left(\partial_\xi^\delta T\cdot\partial_\xi^{\alpha-\delta}f\right).
\end{aligned}$$

So the claim is true for all multi-indices $\alpha\in\mathbb{N}^m$.

2. For a given multi-index $\alpha\in\mathbb{N}^m$ we define a mapping $F_\alpha:\mathbb{R}^m\to\mathbb{R}$, $x\mapsto\prod_{i=1}^m x_i^{\alpha_i}$. By the Taylor Formula and the fact that $\partial^\gamma F\equiv 0$ for each $\gamma\in\mathbb{N}^m$ with $\gamma_i>\alpha_i$ for an index $i\in\{1,\dots,m\}$, we obtain:

$$\begin{aligned}
F_\alpha(x-y) &= \sum_{\gamma\le\alpha}\frac{\prod_{i=1}^m(-y_i)^{\alpha_i}}{\gamma!}\cdot\partial^\gamma F_\alpha(x)+0\\
&= \sum_{\gamma\le\alpha}\frac{1}{\gamma!}\cdot\left(\prod_{i=1}^m(-y_i)^{\alpha_i}\right)\cdot\left(\frac{\alpha!}{(\alpha-\gamma)!}\prod_{j=1}^m x_j^{\alpha_j-\gamma_j}\right).
\end{aligned}$$

This yields, by setting $x=y=(1,\dots,1)\in\mathbb{R}^m$:

$$\frac{\delta_{\alpha,0}}{\alpha!} = \frac{F_\alpha(0)}{\alpha!} = \sum_{\gamma\le\alpha}\frac{1}{\gamma!}\cdot\left(\prod_{i=1}^m(-1)^{\alpha_i}\right)\cdot\left(\frac{1}{(\alpha-\gamma)!}\prod_{j=1}^m 1^{\alpha_j-\gamma_j}\right) = \sum_{\gamma\le\alpha}\frac{(-1)^{|\alpha-\gamma|}}{\gamma!(\alpha-\gamma)!}.$$

□

Theorem 2.98. *Let $\pi:\mathbb{L}\to M$ and $\varpi:\mathbb{E}\to N$ be two bundles of Lie algebras with fiber $\mathfrak{k}$ and $\mathfrak{g}$, respectively,* $\dim\mathfrak{k}=d$, $\dim\mathfrak{g}=e$, $\mathrm{Hom}(\mathfrak{k}/[\mathfrak{k},\mathfrak{k}],\mathfrak{z}(\mathfrak{k}))=0$, $\mathrm{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],\mathfrak{z}(\mathfrak{g}))=0$ *and* $\mathrm{S}(\mathfrak{k})=\mathbb{K}\cdot\mathbf{1}$, $\mathrm{S}(\mathfrak{g})=\mathbb{K}\cdot\mathbf{1}$, *so that both Lie algebras are indecomposable by Lemma 2.28. Suppose there exists an isomorphism of Lie algebras $\mu:\Gamma^k(\mathbb{L})\to\Gamma^\ell(\mathbb{E})$ for some $0\ne k,\ell\in\overline{\mathbb{N}}$. Then $k=\ell$ and*

(a) if $k,\ell\in\mathbb{N}$, then μ is induced in a natural way by a C^k-isomorphism of vector bundles $\kappa:\mathbb{L}\to\mathbb{E}$, i.e., if $\kappa':M\to N$ is the map underlying to κ (that means $\kappa'\circ\pi=\varpi\circ\kappa$), then for all $X\in\Gamma^k(\mathbb{L})$ we have $\mu(X)=\kappa\circ X\circ(\kappa')^{-1}$. In particular, the manifolds M,N are C^k-diffeomorphic and the Lie algebras $\mathfrak{k},\mathfrak{g}$ are isomorphic.

(b) if $k,\ell=\infty$, then the manifolds M,N are diffeomorphic and the Lie algebras $\mathfrak{k},\mathfrak{g}$ are isomorphic. After identifying the manifolds and the Lie algebras, μ turns into a linear differential operator of order at most $e-1$ taking the following local form on a bundle chart (U,φ) of $\pi:\mathbb{L}\to M$ with corresponding chart (U,ξ) of M:

$$A^\varphi \overset{\mu^\varphi}{\longmapsto} \sum_{|\alpha|<d}\frac{1}{\alpha!}N^\alpha\cdot\left(\mu_0\cdot\partial_\xi^\alpha A^\varphi\right), \tag{20}$$

where μ_0 is a smooth function $U\to\mathrm{Aut}(\mathfrak{g})$ and we use the notation $N^\alpha=N_1^{\alpha_1}\circ\ldots\circ N_m^{\alpha_m}$ for multi-indices $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m$ and smooth functions $N_1,\dots,N_m\in\mathrm{C}^\infty(U,\mathrm{N}(\mathfrak{g}))$.

Proof. The isomorphism of Lie algebras $\mu : \Gamma^k(\mathbb{L}) \to \Gamma^\ell(\mathbb{E})$ induces an isomorphism of associative algebras $\widetilde{\mu} : \mathrm{Cent}\left(\Gamma^k(\mathbb{L})\right) \to \mathrm{Cent}\left(\Gamma^\ell(\mathbb{E})\right)$ by $\widetilde{\mu}(T) := \mu \circ T \circ \mu^{-1}$ for $T \in \mathrm{Cent}\left(\Gamma^k(\mathbb{L})\right)$. This identifies, by Theorem 2.86, with an isomorphism $\Gamma^k(\mathrm{Cent}(\mathbb{L})) \to \Gamma^\ell(\mathrm{Cent}(\mathbb{E}))$ and, by applying Lemma 2.18.3 to the fibers of $\mathrm{Cent}(\mathbb{E})$, we deduce that $\widetilde{\mu}$ induces an isomorphism of associative algebras $v : \mathrm{C}^k(M, \mathbb{K}) \to \mathrm{C}^\ell(N, \mathbb{K})$ as follows: By (17), an arbitary element of $\mathrm{S}\left(\Gamma^k(\mathbb{L})\right)$ takes the form $f \cdot \mathbf{1}$, where $f \in \mathrm{C}^k(M, \mathbb{K})$. By another application of (17), we see that there is a function $v(f) \in \mathrm{C}^\ell(N, \mathbb{K})$ such that we have

$$\widetilde{\mu}(f \cdot \mathbf{1}) = v(f) \cdot \mathbf{1} + N_f, \tag{21}$$

where $N_f \in \mathrm{N}\left(\Gamma^\ell(\mathbb{E})\right) = \Gamma^\ell(\mathrm{N}(\mathbb{E}))$ is nilpotent and this decomposition is unique. Note that for constant functions $f \equiv c$, the map $\widetilde{\mu}(f \cdot \mathbf{1}) = c \cdot \widetilde{\mu}(\mathbf{1}) = c \cdot \mathbf{1}$ is semisimple and thus, by uniqueness of the decomposition (21), we have $N_f = 0$ for constant functions f. The morphism property of v is shown by the following calculations:

$$\widetilde{\mu}(f \cdot \mathbf{1}) \cdot \widetilde{\mu}(g \cdot \mathbf{1}) = \widetilde{\mu}((f \cdot \mathbf{1}) \cdot (g \cdot \mathbf{1})) = \widetilde{\mu}(fg \cdot \mathbf{1}) = v(fg) \cdot \mathbf{1} + N_{fg}$$

and

$$\widetilde{\mu}(f \cdot \mathbf{1}) \cdot \widetilde{\mu}(g \cdot \mathbf{1}) = (v(f) \cdot \mathbf{1} + N_f) \cdot (v(g) \cdot \mathbf{1} + N_g) = v(f)v(g) \cdot \mathbf{1} + \underbrace{v(f) \cdot N_g + v(g) \cdot N_f + N_f N_g}_{\text{nilpotent}}.$$

Furthermore, the Binomial Theorem yields, for $f \in \mathrm{C}^k(M, \mathbb{K})$ and $r \in \mathbb{N}$:

$$N_{f^r} = \sum_{t=1}^{r} \binom{r}{t} v(f)^{r-t} (N_f)^t .$$

Note that v is bijective because $\widetilde{\mu}_{|\,\mathrm{S}(\Gamma^k(\mathbb{L}))}$ and $\widetilde{\mu}^{-1}_{|\,\mathrm{S}(\Gamma^k(\mathbb{E}))}$ are injective. By applying Lemma 2.96 to $v : \mathrm{C}^k(M, \mathbb{K}) \to \mathrm{C}^\ell(N, \mathbb{K})$, we obtain $k = \ell$ and the existence of a C^k-diffeomorphism $\lambda : M \to N$ such that $v(f) = f \circ \lambda^{-1}$ for all $f \in \mathrm{C}^k(M, \mathbb{K})$. We now identifiy the manifolds M, N via λ and the associative algebras $\mathrm{C}^k(M, \mathbb{K})$, $\mathrm{C}^k(N, \mathbb{K})$ via v, so that we have the isomorphisms $\mu : \Gamma^k(\mathbb{L}) \to \Gamma^k(\mathbb{E})$ and $\widetilde{\mu} : \Gamma^k(\mathrm{Cent}(\mathbb{L})) = \mathrm{Cent}\left(\Gamma^k(\mathbb{L})\right) \longrightarrow \Gamma^k(\mathrm{Cent}(\mathbb{E})) = \mathrm{Cent}\left(\Gamma^k(\mathbb{E})\right)$. For all sections $A \in \Gamma^k(\mathbb{L})$, all points $x \in M$ and all mappings $f \in \mathrm{C}^k(M, \mathbb{K})$ with $f(x) = 0$, we calculate:

$$\begin{aligned}(\mu(f^e A))_x &= (\mu((f^e \cdot \mathbf{1})(A)))_x = ((\widetilde{\mu}(f^e \cdot \mathbf{1}))(\mu(A)))_x = ((f^e \cdot \mathbf{1} + N_{f^e})(\mu(A)))_x \\ &= (f^e \mu(A))_x + (N_{f^e}(\mu(A)))_x = f(x)^e (\mu(A))_x + \sum_{t=1}^{e} \binom{e}{t} f(x)^{e-t} (N_f)^t_x (\mu(A))_x = 0 + 0 = 0.\end{aligned} \tag{22}$$

Suppose a section $A \in \Gamma^k(\mathbb{L})$ be zero on an open set $U \subseteq M$, $x \in U$ and let $\rho : M \to [0,1]$ be a smooth function and $W \subseteq U$ a smaller x-neighbourhood such that $\rho_{|M \setminus U} \equiv 0$ and $\rho_{|W} \equiv 1$. Then $A = (1-\rho)^e A$ and (22) shows $(\mu(A))_x = 0$. Since x was arbitrarily chosen, $\mu(A)$ is also zero on U. So $\mu : \Gamma^k(\mathbb{L}) \to \Gamma^k(\mathbb{E})$ is local and, by the Peetre Theorems 2.63 and 2.65, a differential operator.

If $k \in \mathbb{N}$, then Theorem 2.65 even implies that μ is a differential operator of order 0 inducing a bundle isomorphism $\kappa : \mathbb{L} \to \mathbb{E}$ (cf. Definition 2.33 and Remark 2.66). This proves (a).

Now assume $k = \infty$. Let $A \in \Gamma(\mathbb{L})$ be a section with $j^{e-1}_x(A) = 0$. By Lemma 2.46, the section A locally takes the form

$$A = \sum_{i=1}^{r} f_i^e A_i$$

for functions $f_i \in \mathrm{C}^\infty(M, \mathbb{K})$, $f_i(x) = 0$ and $A_i \in \Gamma(\mathbb{L})$. By using relation (22), we see that

$$(\mu(A))_x = \sum_{i=1}^{r} [\mu(f_i^e A_i)]_x = \sum_{i=1}^{r} 0 = 0.$$

This proves that the order of the differential operator μ is at most $e-1$ because $j_x^{e-1}(A)=0$ already implies $(\mu(A))_x=0$. Let (U,φ) be a bundle chart of $\pi:\mathbb{L}\to M$ with corresponding chart (U,ξ) of M such that $\xi(U)$ is convex. We have the local forms $\mu^\varphi:\mathrm{C}^k(U,\mathfrak{k})\to\mathrm{C}^k(U,\mathfrak{g})$, $\widetilde{\mu}^\varphi:\mathrm{C}^k(U,\mathrm{Cent}(\mathfrak{k}))\to\mathrm{C}^k(U,\mathrm{Cent}(\mathfrak{g}))$ and $N_f^\varphi\in\mathrm{C}^k(U,\mathrm{N}(\mathfrak{g}))$ of μ, $\widetilde{\mu}$ and $N_f\in\Gamma^k(\mathrm{N}(\mathbb{E}))$, respectively. Locally, the decomposition (21), turns into:

$$\widetilde{\mu}^\varphi(f\cdot\mathbf{1})=f\cdot\mathbf{1}+N_f^\varphi.$$

The Taylor Formula yields, for each $x\in U$ and $A\in\Gamma(\mathbb{L})$:

$$j_x^{e-1}\left(y\longmapsto A^\varphi(y)-\sum_{|\alpha|<e}\frac{\prod_{i=1}^m(\xi_i(y)-\xi_i(x))^{\alpha_i}}{\alpha!}\cdot\partial_\xi^\alpha A^\varphi(x)\right)=0. \tag{23}$$

For $x,y\in U$, $i\in\{1,\dots,m\}$ and $\alpha\in\mathbb{N}^m$ with $|\alpha|<e$ we write:

$$\begin{aligned}\xi_{i,x}(y)&:=\xi_i(y)-\xi_i(x),\\ \Xi_{\alpha,x}(y)&:=\prod_{i=1}^m(\xi_i(y)-\xi_i(x))^{\alpha_i},\\ \phi_{\alpha,x}(y)&:=\prod_{i=1}^m(\xi_i(y)-\xi_i(x))^{\alpha_i}\cdot\partial_\xi^\alpha A^\varphi(x).\end{aligned}$$

Then we define smooth mappings $N_1,\dots,N_m:U\to\mathrm{N}(\mathfrak{g})$ and $\mu_0:U\to\mathrm{Hom}(\mathfrak{k},\mathfrak{g})$ (in the sense of Lie algebra morphisms), by setting for $x\in U$, $u\in\mathfrak{k}$ and the constant map $c_u:U\to\mathfrak{g}$, $y\mapsto u$:

$$\mu_0(x)(u):=\mu^\varphi(c_u)(x)$$

and

$$\begin{aligned}N_i(x)&:=N^\varphi_{\xi_{i,x}}(x)=\widetilde{\mu}^\varphi(\xi_{i,x}\cdot\mathbf{1})(x)-(\xi_{i,x}\cdot\mathbf{1})(x)\\ &=\widetilde{\mu}^\varphi(\xi_{i,x}\cdot\mathbf{1})(x)\end{aligned}$$

We calculate:

$$\begin{aligned}\mu^\varphi(\phi_{\alpha,x})&=\mu^\varphi\left(y\mapsto\left(\prod_{i=1}^m(\xi_i(y)-\xi_i(x))^{\alpha_i}\cdot\mathbf{1}\right)\left(\partial_\xi^\alpha A^\varphi(x)\right)\right)\\ &=\widetilde{\mu}^\varphi(\Xi_{\alpha,x}\cdot\mathbf{1})\cdot\mu^\varphi\left(y\mapsto\partial_\xi^\alpha A^\varphi(x)\right)\\ &=\left(\prod_{i=1}^m\underbrace{\widetilde{\mu}^\varphi\left(\xi_{i,x}^{\alpha_i}\cdot\mathbf{1}\right)}_{=N_i^{\alpha_i}}\right)\cdot\left(\mu_0\cdot\partial_\xi^\alpha A^\varphi\right)\\ &=\left(\prod_{i=1}^m N_i^{\alpha_i}\right)\cdot\left(\mu_0\cdot\partial_\xi^\alpha A^\varphi\right)=N^\alpha\cdot\left(\mu_0\cdot\partial_\xi^\alpha A^\varphi\right).\end{aligned}$$

Thus, by (23) and the fact that μ is of order at most $e-1$, we have the local form

$$A^\varphi\xmapsto{\mu^\varphi}\sum_{|\alpha|<e}\frac{1}{\alpha!}N^\alpha\cdot\left(\mu_0\cdot\partial_\xi^\alpha A^\varphi\right). \tag{24}$$

It remains to show that each $\mu_0(x)$, where $x \in U$, is bijective. This will be done in two steps. First, we verify the following identitiy for $A \in \Gamma(\mathbb{L})$ and $B \in \Gamma(\mathbb{E})$ with $\mu(A) = B$ (implying $\mu^\varphi(A^\varphi) = B^\varphi$):

$$P(A) := \sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha \cdot (\mu_0 \cdot A^\varphi)\right) = \sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha \cdot B^\varphi\right) =: Q(B). \tag{25}$$

Note:

$$\text{If } S_1, \ldots, S_m \in \mathrm{N}(\mathfrak{g}) \text{ and } |\alpha| \geq e, \text{ then } S^\alpha = S_1^{\alpha_1} \circ \ldots \circ S_m^{\alpha_m} = 0 \tag{26}$$

because S^α can be written as a sum of $|\alpha|$-th powers of linear combinations of the S_i and the e-th power of a nilpotent morphism cotained in $\mathrm{N}(\mathfrak{g})$ is zero. Therefore we can perform the following manipulations:

$$\begin{aligned}
\sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha \cdot B^\varphi\right) &\overset{(24)}{=} \sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha \cdot \sum_{|\beta|<e} \frac{1}{\beta!} N^\beta \cdot \left(\mu_0 \cdot \partial_\xi^\beta A^\varphi\right)\right) \\
&= \sum_{\substack{|\alpha|<e \\ |\beta|<e}} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial_\xi^\alpha \left(N^{\alpha+\beta} \cdot \left(\mu_0 \cdot \partial_\xi^\beta A^\varphi\right)\right) \\
&\overset{(26)}{=} \sum_{|\alpha+\beta|<e} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial_\xi^\alpha \left(\left(N^{\alpha+\beta} \circ \mu_0\right) \cdot \partial_\xi^\beta A^\varphi\right) \\
&\overset{(18)}{=} \sum_{\substack{|\alpha+\beta|<e \\ \gamma \leq \alpha}} \frac{(-1)^{|\alpha|}}{\beta!\gamma!(\alpha-\gamma)!} \partial_\xi^\gamma \left(N^{\alpha+\beta} \circ \mu_0\right) \cdot \partial_\xi^{\alpha+\beta-\gamma} A^\varphi.
\end{aligned}$$

Now we perform the following substitutions: $\alpha' := \alpha + \beta - \gamma$, $\beta' := \gamma$ and $\gamma' := \beta$, thus $\alpha + \beta = \alpha' + \beta'$, $\alpha - \gamma = \alpha' - \gamma'$ and $\gamma \leq \alpha \iff \gamma' \leq \alpha'$. And so we can finally show (25):

$$\begin{aligned}
\sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha \cdot B^\varphi\right) &= \sum_{\substack{|\alpha+\beta|<e \\ \gamma \leq \alpha}} \frac{(-1)^{|\alpha|}}{\beta!\gamma!(\alpha-\gamma)!} \partial_\xi^\gamma \left(N^{\alpha+\beta} \circ \mu_0\right) \cdot \partial_\xi^{\alpha+\beta-\gamma} A^\varphi \\
&= \sum_{\substack{|\alpha'+\beta'|<e \\ \gamma' \leq \alpha'}} \frac{(-1)^{|\alpha'-\gamma'|} \cdot (-1)^{\beta'}}{\gamma'!\beta'!(\alpha'-\gamma')!} \partial_\xi^{\beta'} \left(N^{\alpha'+\beta'} \circ \mu_0\right) \cdot \partial_\xi^{\alpha'} A^\varphi \\
&= \sum_{|\alpha'+\beta'|<e} \left(\sum_{\gamma' \leq \alpha'} \frac{(-1)^{|\alpha'-\gamma'|}}{\gamma'!(\alpha'-\gamma')!}\right) \cdot \frac{(-1)^{\beta'}}{\beta'!} \partial_\xi^{\beta'} \left(N^{\alpha'+\beta'} \circ \mu_0\right) \cdot \partial_\xi^{\alpha'} A^\varphi \\
&\overset{(19)}{=} \sum_{|\beta'|<e} \frac{(-1)^{|\beta'|}}{\beta'!} \partial_\xi^{\beta'} \left(N^{\beta'} \cdot (\mu_0 \cdot A^\varphi)\right).
\end{aligned}$$

By the definition of P and μ_0, if $A^\varphi(x) = u = A'^\varphi(x)$ for $A, A' \in \Gamma(\mathbb{L})$ and $x \in U$, then:

$$P(A)(x) = \sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha(x) \left(\mu_0(x) \left(A^\varphi(x)\right)\right)\right) = \sum_{|\alpha|<e} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \left(N^\alpha(x) \left(\mu^\varphi(c_u)(x)\right)\right) = P(A')(x).$$

So $P(A)(x) \in \mathfrak{g}$ depends on A^φ only via $A^\varphi(x) \in \mathfrak{k}$ and we can define a linear map $P_x : \mathfrak{k} \to \mathfrak{g}$ by setting $P_x(A^\varphi(x)) := P(A)(x)$ for $x \in U$. We may also define linear maps $Q_x : \mathfrak{g} \to \mathfrak{g}$ for $x \in U$ by $Q_x(B^\varphi(x)) := Q(B)(x)$ for $x \in U$ due to an analogous element as above with the N^α instead of μ_0. Since any Q_x is a sum of $\mathbf{1}$ and a nilpotent linear map (see the sum in the third term of (25) evaluated in x),

it is bijective. We define a smooth mapping $\eta_0 : U \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{k})$ (in the sense of Lie algebra morphisms) by setting for $x \in U$, $v \in \mathfrak{g}$ and the constant map $c_v : U \to \mathfrak{g}$, $x' \to v$:

$$\eta_0(x)(v) := \left(\mu^{-1}\right)^{\varphi}(c_v)(x).$$

We now fix $x \in U$ and $v \in \mathfrak{g}$. Since μ is surjective, there exists $A \in \Gamma(\mathbb{L})$ such that $\mu^{\varphi}(A^{\varphi}) = c_v$, thus $A^{\varphi}(x) = \eta_0(x)(v)$. By (25), we have $P_x(\eta_0(x)(v)) = Q_x(v)$. The injectivity of Q_x implies the injectivity of $\eta_0(x) : \mathfrak{g} \to \mathfrak{k}$, yielding $e \leq d$. By the symmetry of the arguments, $\mu_0(x) : \mathfrak{k} \to \mathfrak{g}$ is also injective, yielding $d \leq e$. So $\mu_0(x)$ is an isomorphism of Lie algebras. □

Remark 2.99. If the Lie algebra $\mathfrak{k}$ is complex simple, then it is central, i.e. $\mathrm{Cent}(\mathfrak{k}) = \mathbb{C} \cdot \mathbf{1}$, by the Schur Lemma. If the Lie algebra $\mathfrak{k}$ is real simple, then Proposition X.1.5 of [He78] says that $\mathfrak{k}$ satisfies exactly one of the following two conditions:

A. $\mathfrak{k}$ admits a complex structure and the complexification $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is the direct sum of two simple isomorphic ideals, hence $\mathfrak{k}_{\mathbb{C}}$ is not a simple $\mathbb{C}$-Lie algebra.

B. $\mathfrak{k}_{\mathbb{C}}$ is a simple $\mathbb{C}$-Lie algebra.

If the real simple Lie algebra $\mathfrak{k}$ is the Lie algebra associated to a compact Lie group, then its complexificaton $\mathfrak{k}_{\mathbb{C}}$ is complex simple by Lemma X.1.3 of [He78], so we are in case B and have $\mathrm{Cent}(\mathfrak{k}) = \mathbb{R} \cdot \mathbf{1}$.

Corollary 2.100. *If we have one of the following two cases:*

1. *the Lie algebra $\mathfrak{k}$ is complex simple,*

2. *the Lie algebra $\mathfrak{k}$ is real simple and associated to a compact Lie group,*

then for any $\mu \in \mathrm{Aut}\left(\Gamma^k(\mathbb{L})\right)$, where $k \in \overline{\mathbb{N}}$, there is a C^k-diffeomorphism $\lambda : M \to M$ such that μ can be identified with some $\mu_0 \in \Gamma^k(\mathrm{Aut}_\lambda(\mathbb{L}))$, i.e. a C^k-section in $\Gamma^k(\mathrm{Hom}(\mathbb{L}, \mathbb{L}))$ where for all $x \in M$ the map $\mu_0(x) : \mathbb{L}_x \to \mathbb{L}_{\lambda(x)}$ is an isomorphism of Lie algebras. The bundle $\mathrm{Aut}_\lambda(\mathbb{L})$ is isomorphic to $\mathrm{Aut}(\mathbb{L}) := \mathrm{Aut}_{\mathrm{id}_M}(\mathbb{L})$ by $\left(f : \mathbb{L}_x \to \mathbb{L}_{\lambda(x)}\right) \longmapsto \left(\mu^{-1} \circ f : \mathbb{L}_x \to \mathbb{L}_x\right)$.

Corollary 2.101. *In both of the cases of Corollary 2.100, the Lie algebra of C^k-sections, where $k \in \overline{\mathbb{N}}$, of the trivial bundle $\mathbb{L} = M \times \mathfrak{k}$ is naturally isomorphic to $\mathrm{C}^k(M, \mathfrak{k})$ and, since $\mathrm{Diff}^k(M)$ and $\Gamma^k(\mathrm{Aut}(\mathbb{L})) \cong \mathrm{C}^k(M, \mathrm{Aut}(\mathfrak{k}))$ can be naturally embedded into $\mathrm{Aut}(\mathrm{C}^k(M, \mathfrak{k}))$ as a subgroup and a normal subgroup, respectively, we obtain the isomorphism*

$$\mathrm{Aut}(\mathrm{C}^k(M, \mathfrak{k})) \cong \mathrm{C}^k(M, \mathrm{Aut}(\mathfrak{k})) \rtimes \mathrm{Diff}^k(M).$$

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